Control of fed-batch bioreactors models by means of dynamic time-scale transformation and operatorial parametrization

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Abstract: In this paper, we show how operatorial transformations, defined in the trajectorial sense, can be used to significantly simplify the problem of control of fed-batch bioreactors. We first define the suitable mathematical framework and study remarkable transformations. We also introduce the notion of operatorial parametrizing and apply these notions to the control of fed-batch bioreactors models.

Keywords: Operatorial transformation, Fed-batch Bioreactor, Operatorial Parametrizing, Time Scale Transformation, Nonlinear Control.

1. INTRODUCTION

Control of fermentation processes is a problematic which has been intensively studied during the last decades. The complexity of biological underlying phenomena make this problematic tricky due to strong nonlinearity and uncertainties of the models.

Various approaches have been developped; some are based on predictive quality of the models Winkin et al. [2009], Moya et al. [2002], Peroni et al. [2005], Rani and Rao [1999]; others are mainly based on fuzzy networks Ronen et al. [2002], Zhihua and Jie [2002].

In this paper, we show how *operatorial transformations*, viewed as function of trajectories, can be used to significantly simplify the problem of control of fed-batch bioreactors. We first define the suitable framework for those transformations. Then, the time-scale transformations are introduced and a simple and efficient strategy for decoupling state/input components is stated. In the third section, we define the operatorial parametrizing of dynamic systems and problems. Finally, those transformations are applied in the last section to control a class of fed-batch bioreactors models.

2. OPERATORIAL TRANSFORMATIONS

We summarize in this section some basic notions used in the sequel. We namely propose an abstract formulation of dynamic problems, well adapted to the problematic of operatorial transformations, in a generic framework.

We call *trajectory* any function x defined on a time interval $[0, T]$ with values in a topological vector (usually Banach) space $\mathbf{X}$.

We consider *dynamic problems* of the abstract form:

$$\begin{cases} \mathbf{\Phi}(u, x) = 0 \\ \mathbb{P}(u, x) = 0, \end{cases} \quad (u, x) \in \mathcal{U} \times \mathcal{X}, \tag{1}$$

where $\mathbf{\Phi} = 0$ is a dynamic model supposed to be well-posed and $\mathbb{P}$ is a "property" that would be satisfied by (u, x), with the convention: $0 \sim$True and $1 \sim$False [1].

This abstract formulation can include most of classical dynamic problems encountered in practice, such as estimation, identification Casenave [2010], control, analysis, numerical simulation etc. Montseny [2009].

Example 1. Classical differential systems:

$$\begin{cases} \partial_t X = f(u, x),\ t \in]0, T[\\ x(0) = x_0, \end{cases} \tag{2}$$

are of the form $\mathbf{\Phi}(u, x) = 0$ with:

$$\mathbf{\Phi} := \begin{pmatrix} \partial_t - f(u, x) \\ < \delta, \cdot > -x_0 \end{pmatrix}, \tag{3}$$

where $< \delta, \cdot >$ is the Dirac operator $v \mapsto v(0)$. Manifolds $\mathcal{U}$ and $\mathcal{X}$ must be adapted to the problem, for example $\mathcal{U} \subset L^\infty(0, T),\ \mathcal{X} \subset C^0([0, T])$.

Then, we call *operatorial transformation* any operator that transforms a problem $(\mathbf{\Phi}, \mathbb{P}) = 0$ on $\mathcal{U} \times \mathcal{X}$ into a "new" problem $(\tilde{\mathbf{\Phi}}, \tilde{\mathbb{P}}) = 0$ on $\tilde{\mathcal{U}} \times \tilde{\mathcal{X}}$, such that:

$$\tilde{\mathbf{\Phi}}(\tilde{u}, \tilde{x}) = 0 \Rightarrow \mathbf{\Phi}(u, x) = 0, \tag{4}$$

that is the solution of the initial problem is deduced from the solution of the "new" problem.

3. TIME-SCALE TRANSFORMATIONS

There exists many examples of dynamic systems for which it can be defined an intrinsic clock, under which the dynamic equations are greatly simplified (see for example Fangtang and Kelkari [2003], Bliman and Sorine [1996]). A generic formulation of such "time-scale transformations "(TST) reveals several interesting properties. We only give here some essential results relating to those transformations; more properties, results and extensions can be found in Montseny [2010, 2009].

[1] Namely, the property $\mathbb{P}$ could have the expression $\mathbf{J}(u, x) = 0$ with $\mathbf{J}$ a continuous operator, or $\mathcal{J}(u, x) = \min$ with $\mathcal{J}$ a cost functional.

In the following, $\mathbf{X}$ is a Banach space and $\mathfrak{X}$ a suitable space of trajectories with values in $\mathbf{X}$, and ∂_t^{-1} denotes the integration operator: $u \mapsto \int_0^t u$.

The symbol $(\cdot)'$ represents the differentiation operation.

3.1 Definitions and properties

Basically, a TST is a trajectorial transformation of the form $x \mapsto x \circ \varphi^{-1}$ where the strictly increasing[2] function $\varphi(t)$ defines a new time-scale $\tau := \varphi(t)$.

Definition 2. The TST operator $\mathbf{S}$ on $\mathfrak{X}$ is by:

$$(x, \varphi) \mapsto \mathbf{S}(x, \varphi) := x \circ \varphi^{-1}. \tag{5}$$

For convenience, we denote:

- $\mathbf{S}_\varphi := \mathbf{S}(\cdot, \varphi)$.
- $\tau := \varphi(t)$ the "new time" and $\widetilde{x}$ the so-transformed trajectory x, that is:
$$\widetilde{x} := \mathbf{S}_\varphi(x) = x \circ \varphi^{-1}.$$

The inversion of a time-scale transformation, essential to keep equivalence of models, is simply given by the relation:

$$\mathbf{S}_\varphi^{-1} = \mathbf{S}_{\varphi^{-1}}.$$

Definition 3. A time-scale-transformation is said *dynamic* if the clock φ is the result of a dynamic transformation of a function v, that is $\varphi = \boldsymbol{\varphi}(v)$ with $\boldsymbol{\varphi}$ a causal operator on a manifold $\mathcal{V}$ (of trajectories), such that $\forall v \in \mathcal{V}$, $\boldsymbol{\varphi}(v)$ is continuous and strictly increasing.

We denote $\mathbf{S}_{\boldsymbol{\varphi}}$ the operatorial function:

$$\mathbf{S}_{\boldsymbol{\varphi}} : v \mapsto \mathbf{S}_{\boldsymbol{\varphi}(v)}.$$

Remark 4. A dynamic time-scale transformation can be applied on the trajectory v itself :

$$\widetilde{v} = \mathbf{S}_{\boldsymbol{\varphi}(v)}(v) = v \circ \boldsymbol{\varphi}(v)^{-1}.$$

Note that because the operator $\boldsymbol{\varphi}$ is causal, this expression will be compatible with real time applications.

An important example of causal dynamic TST operator is given by $\boldsymbol{\varphi} = \partial_t^{-1}$. In particular, we have the following proposition, on which is based the singularity simplification detailed on the next paragraph.

Proposition 5. Let g a continuous and strictly positive function and x differentiable. Then:

$$\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}(g\, x') = [\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}(x)]'. \tag{6}$$

Roughly speaking:

$$\mathbf{S}_{\partial_t^{-1}\frac{1}{g}} : \; g\,\partial_t \mapsto \partial_\tau. \tag{7}$$

Proof. We denote $\varphi := \partial_t^{-1}\frac{1}{g}$. Using the chain rule, we have:

$$\mathbf{S}_\varphi(g\,x') = \mathbf{S}_\varphi(g)\;\mathbf{S}_\varphi(x') = \widetilde{g}\,\widetilde{\varphi'}\;\widetilde{x}' = \widetilde{g}\,\frac{1}{\widetilde{g}}\,\widetilde{x}' = \widetilde{x}'.$$

■

Note that g that can be any function of time, for exemple of the form $G(t, u, x)$.

[2] However, the use of non invertible time-scale transformations is not excluded, namelly when dealing with models involving dry friction, see for example Montseny [2009].

3.2 Decoupling of state/input components in dynamical models

Transformation of differential models We consider an abstract dynamic model of the form:

$$\mathbf{\Phi}(u, x, g\,\partial_t x) = 0, \tag{8}$$

with g a continuous and stricly positive function.

Corollary 6. By the time-scale transformation $\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}$, equation (8) is transformed into:

$$\mathbf{\Phi}(\tilde{u}, \widetilde{x}, \partial_\tau \widetilde{x}) = 0, \tag{9}$$

the correspondence between τ and t being indifferently defined by $\partial_t \tau = \frac{1}{g}$ or $\partial_\tau t = \widetilde{g}$.

Thus, a suitable TST transforms a first order differential equation $g\,\partial_t x = F(u, x)$ into the following time-invariant differential equation:

$$\partial_\tau \widetilde{x} = F(\tilde{u}, \tilde{x}).$$

In other words, such transformation allow to suppress some undesirable terms of a model by "absorbing" them into the new time derivative operator ∂_τ. Then, the resolution of dynamic problems on such models can be simplified.

Decoupling of state/input components Let us consider the model:

$$\partial_t x = f(x) + g(u)\,h(x) \tag{10}$$

with $h(x)$ a positive function. Then, thanks to proposition 6, the dynamic time-scale transformation $\mathbf{S}_{\partial_t^{-1} h(x)}$ leads to the model:

$$\partial_\tau \widetilde{X} = \frac{f(\widetilde{x})}{h(\widetilde{x})} + g\,(\widetilde{u})\,, \tag{11}$$

the correspondance between τ et t being indifferently defined by $\partial_t \tau = h(x)$ or $\partial_\tau t = \dfrac{1}{h\,(\widetilde{x})}$. The system (11) (in time τ) presents the advantage to have an additive and decoupled input $g\,(\widetilde{u})$.

The same transformation can of course be operated on $g(u)$ (if g is a positive function), to isolate the term $h(x)$ by using the time-scale transformation $\partial_t^{-1}(g(u))$. This has been used for the problem of bioreactor control presented in section 5. Another application of decoupling by dynamic TST can be found in Montseny and Camon [2010].

Remark 7. The above results can be extended to nonlocal dynamic models Montseny [2009].

Remark 8. Decoupling of state/input components is not the only simplificiations brought by proposition 6. We can for example mention the desingularization of nonlocal dynamic models of the form:

$$H(\partial_t)\,x = \frac{1}{h(t, u, x)} F(t, u, x),$$

with $h(t, u, x)$ close to zero, or even the transformation of differential inclusions into classical differential equations. One can refer to Montseny [2010, 2009] for some applications.

4. OPERATORIAL PARAMETRIZING OF DYNAMIC PROBLEMS

We recall the abstract expression of dynamic problems on $\mathcal{U} \times \mathcal{X}$:

$$\mathbf{\Phi}(u,x) = 0 \tag{12a}$$
$$\mathbb{P}(u,x) = 0. \tag{12b}$$

In a synthetic way, the aim of parametrizing is to express solutions of the model $\mathbf{\Phi}(u,x) = 0$ from a quantity y with a relation of the form $(u,x) = \mathbf{Q}(y)$. Many works are related to this problematic Jakubvzyk and Respondek [1980], Fujimoto and Sugie [1996]. Other works focused on algebraic properties of the problem in a general formal framework for linear models Chyzak and Robertz [2005]; the specific case of linear partial differential equations were studied in Nihtila et al. [2004].

In this work, we deal with "operatorial"parametrizations, which are globally defined as operators on trajectorial manifolds.

4.1 Parametrizing

In the sequel, we denote $\mathcal{Z} := \mathcal{U} \times \mathcal{X}$, $z := (u,x)$ and $\mathcal{E} \subset \mathcal{U} \times \mathcal{X}$ the (trajectory) manifold of solutions of the dynamic model (12a).

Definition 9. We call *parametrizing* of (12a) a manifold $\mathcal{Y}$ and a continuous operator $\mathbf{Q} : \mathcal{Y} \to \mathcal{Z}$ such that:
$$\forall y \in \mathcal{Y},\ z = \mathbf{Q}(y) \Rightarrow \mathbf{\Phi}(z) = 0, \tag{13}$$
that is $\mathbf{Q}(y)$ is solution of (12a).

The trajectory y is called *parameter*.

Remark 10. The continuity of operators involved in the parametrization process is essential to ensure the practicability of the method and their robustness towards noise and errors.

Then, a parametrizing appear to be a correspondence between solutions of the model (12a) and elements of a manifold $\mathcal{Y}$, leading to a transformation of the whole dynamic problem (12) into:
$$\widetilde{\mathbb{P}}(y) := \mathbb{P}(\mathbf{Q}(y)) = 0,\ y \in \mathcal{Y}, \tag{14}$$
this new problem being equivalent [3] to (12).

The operator $\mathbf{Q}$ allow to deduce, from solution y^* of (14), the related couple (u^*, x^*) solution of (12), without solving the equation (12a); in other words, the manifold "summerizes "the dynamic model (12a), which is no longer necessary to solve for the resolution of the global dynamic problem.

A parametrizing will be especially interesting when the effective resolution of (14) is simpler than the resolution of the initial problem (12).

4.2 Parametric output operators

In practice, a parametrizing is often built by "exctracting" the parameter y from the model $\mathbf{\Phi}(u,x) = 0$, as a function of u et x. This is the aim the following definition.

Definition 11. We call *parametric output operator* of (12a) a continuous application $\mathbf{A} : \mathcal{U} \times \mathcal{X} \to \mathcal{Y}$ such that $\mathbf{A}|_{\mathcal{E}}$ is an homeomorphism.

[3] When all solutions of $\mathbf{\Phi} = 0$ are parameterized; if not, that means that we have (14) $\Rightarrow$ (12): the new problem have less solutions than $\mathbb{P} = 0$.

From this point of view, y is called *parametric output* of the model.

We have:

Proposition 12. If $\mathbf{A}$ is a parametric output operator of (12a), then $(\mathbf{A}|_{\mathcal{E}}^{-1}, \mathcal{Y})$ is a parametrizing of (12a).

Proof. If $(\mathbf{A}, \mathcal{Y})$ is a parametric output operator of (12a), then by definition, $\mathbf{A}|_{\mathcal{E}}$ is an homeomorphism between $\mathcal{E}$ and $\mathcal{Y}$; so, its inverse is a parametrizing. ■

We denote $(\mathbf{B}, \mathbf{C}) := \mathbf{A}|_{\mathcal{E}}^{-1}$ the parametrizing operator defined by a parametric output operator $\mathbf{A}$. We can remark that the relation $(u,x) = (\mathbf{B}(y), \mathbf{C}(y))$ is composed from two decoupled equations, one between the parameter y and the state x, and the other between y and input u. In that sense the (dynamic) equation
$$x = \mathbf{C}(y) \tag{15}$$
can be interpreted as a new (solved) model with input y on which dynamic problems mainly related to the state can by posed (e.g. control problems).

4.3 Parametric equations

It can be usefull (as for example in section 5) to consider a weaker definition of parametrizing, which will not be explicit (i.e. solved) like $z = \mathbf{Q}(y)$ in definition 9, but implicit. This is the aim of the following definition.

Definition 13. Let $\mathcal{Y}$ be a manifold, $\mathcal{P}$ a vector topological space and $\mathbf{\Psi} : \mathcal{Y} \times \mathcal{Z} \to \mathcal{P}$ an operator. The equation in $\mathcal{Y} \times \mathcal{Z}$:
$$\mathbf{\Psi}(y,z) = 0, \tag{16}$$
is said *parametric equation* for (12a) if:
$$\forall\,(y,z) \in \mathcal{Y} \times \mathcal{Z},\ \mathbf{\Psi}(y,z) = 0 \Rightarrow \mathbf{\Phi}(z) = 0. \tag{17}$$

Thus, a parametric equation *implicitly* defines a parametrizing $(\mathbf{Q}, \mathcal{Y})$. The parametric equation can be interpreted as a new model of input y and state $z = (u,x)$; in that sense, this is a state augmentation.

The following proposition is usefull for practical construction of parametrizing. It states that if we know a parametrizing of a model, then we can easily access to a parametrizing of any homeomorphic transformation of the model; this property will be in particular used in section 5.

Proposition 14. Let $\mathbf{\Psi} = 0$ be a parametric equation for (12a), and $\mathbf{S} : \mathcal{Z} \to \tilde{\mathcal{Z}}$ an homeomorphic transformation. Then, with $\tilde{\mathbf{\Psi}}(y,\tilde{z}) := \mathbf{\Psi}(y, \mathbf{S}^{-1}(\tilde{z}))$, $\tilde{\mathbf{\Psi}} = 0$ is a parametric equation for the model $\tilde{\mathbf{\Phi}}(\tilde{z}) := \mathbf{\Phi} \circ \mathbf{S}^{-1}(\tilde{z}) = 0$.

With $(\mathbf{Q}, \mathcal{Y})$ the parametrizing implicitly defined by $\mathbf{\Psi} = 0$, the parametrizing defined by $\tilde{\mathbf{\Psi}} = 0$ is $(\mathbf{S} \circ \mathbf{Q}, \mathcal{Y})$.

Proof. We have:
$$\forall\,(y,\tilde{z}) \in \mathcal{Y} \times \tilde{\mathcal{Z}},\ \tilde{\mathbf{\Psi}}(y,\tilde{z}) = 0 \Leftrightarrow \mathbf{\Psi}(y, \underbrace{\mathbf{S}^{-1}(\tilde{z})}_{\in \mathcal{Z}}) = 0$$

As $\mathbf{\Psi} = 0$ is a parametric equation for $\mathbf{\Phi} = 0$, this implies $\mathbf{\Phi}(\mathbf{S}^{-1}(\tilde{z})) = 0$, that is $\tilde{\mathbf{\Phi}} = 0$. So, $\tilde{\mathbf{\Psi}} = 0$ is a parametric equation for $\tilde{\mathbf{\Phi}} = 0$.

The same reasoning applies to the parametrizing. ■

5. APPLICATION TO CONTROL OF A FED-BATCH BIOREACTOR MODEL

5.1 Parameterization of the bioreactor model

We consider the following model of fed-batch bioreactors Wang et al. [2001]:

$$\begin{cases} \partial_t x = \mu(X)\,x - x\,u \\ \partial_t s = -a_1\mu(X)\,x + (s_i - s)\,u \\ \partial_t p = a_2\mu(X)\,x - p\,u \\ X(0) = X_0, \end{cases} \tag{18}$$

where x, s, p are the respective concentrations of biomass, substrate and product, $X = (x, s, p)^T$, μ is the growth rate, s_i the substrate concentration in feed, u (the control) is the dilution of feed and X_0 the initial conditions.

Time-scale transformation of the model The model (18) is of the form (10), with $u > 0$. Then, using results of paragraph 3.2, we know that the dynamic TST $\mathbf{S}_{\partial_t^{-1}u}$ transforms the model (18) into the following model in time τ:

$$\begin{cases} \partial_\tau\,\widetilde{x} = -\widetilde{x} + \frac{\mu(\widetilde{X})\,\widetilde{x}}{\widetilde{u}} \\ \partial_\tau\,\widetilde{s} = -\widetilde{s} + s_i - a_1\frac{\mu(\widetilde{X})\,\widetilde{x}}{\widetilde{u}} \\ \partial_\tau\,\widetilde{p} = -\widetilde{p} + a_2\frac{\mu(\widetilde{X})\,\widetilde{x}}{\widetilde{u}} \\ \widetilde{X}(0) = X_0, \end{cases} \tag{19}$$

the correspondance between τ and t being defined by $\partial_t\tau = u$ or $\partial_\tau t = \frac{1}{\widetilde{u}}$.

Parametric equation for the model The model (19) suggests the definition of the following parametric output:

$$y = \mathbf{A}(\widetilde{u}, \widetilde{X}) := (\tfrac{\mu(\widetilde{X})\widetilde{x}}{\widetilde{u}}, (x_0, s_0, p_0))^T \in \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{D}_{CI} \tag{20}$$

where $\mathcal{Y}_1 = \{f \in L^\infty(I)\,;\, f > 0\}$ and $\mathcal{D}_{CI} \subset \mathbb{R}^3$ is the admissibility domain of initial conditions.

Indeed, the operator $\mathbf{A}$ is continuous and defines a parametrizing of (19) via the parametric equation:

$$\begin{cases} \partial_\tau\widetilde{x} = -\widetilde{x} + y_1 \\ \partial_\tau\widetilde{s} = -\widetilde{s} + s_i - a_1 y_1 \\ \partial_\tau\widetilde{p} = -\widetilde{p} + a_2 y_1 \\ y = \mathbf{A}(\widetilde{u}, \widetilde{X}); \end{cases} \tag{21}$$

one can easily verify that if $\forall y \in \mathcal{Y}$, (u, X_0, X, y) is solution of (21) then (u, X_0, X) issolution of (19). Then, the resolution of (21) leads to the following explicit expression of the parametrizing operator $\mathbf{Q} = (\mathbf{B}, \mathbf{C})$ of (19):

$$\begin{cases} \widetilde{x} = \mathbf{C}_1(y) := (\partial_\tau + 1)^{-1}(y_1) + y_2\,\mathrm{e}^{-(\cdot)} \\ \widetilde{s} = \mathbf{C}_2(y) := (\partial_\tau + 1)^{-1}(s_i - a_1 y_1) + y_3\,\mathrm{e}^{-(\cdot)} \\ \widetilde{p} = \mathbf{C}_3(y) := (\partial_\tau + 1)^{-1}(a_2 y_1) + y_4\,\mathrm{e}^{-(\cdot)} \\ (\widetilde{u}, X_0) = \mathbf{B}(y) := \left(\frac{\mu(\mathbf{C}(y))\,\mathbf{C}_1(y)}{y_1}, y_2, y_3, y_4\right). \end{cases} \tag{22}$$

Finally, using the results of proposition 14, we know that $(\mathbf{S}^{-1}_{\partial_t^{-1}u} \circ \mathbf{Q}, \mathcal{Y})$ defines a parametric equation of the model (18) (in time t):

$$(X, u) = \mathbf{S}^{-1}_{\partial_t^{-1}u} \circ (\mathbf{C}, \mathbf{B})$$

that is:

$$\begin{cases} x = \left[(\partial_\tau + 1)^{-1}(y_1) + y_2\,\mathrm{e}^{-(\cdot)}\right] \circ \partial_t^{-1}u \\ s = \left[(\partial_\tau + 1)^{-1}(s_i - a_1 y_1) + y_3\,\mathrm{e}^{-(\cdot)}\right] \circ \partial_t^{-1}u \\ p = \left[(\partial_\tau + 1)^{-1}(a_2 y_1) + y_4\,\mathrm{e}^{-(\cdot)}\right] \circ \partial_t^{-1}u \\ (u, X_0) = \left(\frac{\mu(\mathbf{C}(y))\,\mathbf{C}_1(y)}{y_1} \circ \partial_t^{-1}u\,, y_2, y_3, y_4\right). \end{cases} \tag{23}$$

We can see that the parametric equation (21) contains in particular a system of *linear and decoupled differential equations* between the state $\widetilde{x}$ and the parameter y, which is the input of this new model. Then, controls problem can be investigated on those equivalent states equations using classical control techniques of linear systems, and solutions (u, X) of the initial nonlinear model will then be easily deduced from the relation (23).

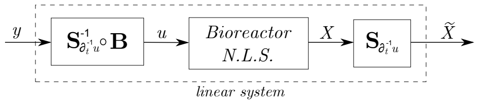

Fig. 1. parametrizing of the nonlinear model.

5.2 Control of the parameterized model

Biomass control Thanks to the decoupled and linear structure of the parametric equation, control of the biomass equation:

$$\begin{cases} \partial_\tau\widetilde{x} = -\widetilde{x} + y_1 \\ \widetilde{x}\,(0) = x_0, \end{cases} . \tag{24}$$

can be easily processed with a classical corrector of proportionnal-integral type:

$$K(p) = \left(1 + \tfrac{1}{T_i\,p}\right)K. \tag{25}$$

We then deduce the related *nonlinear dynamic corrector* decribed by. fig.2.

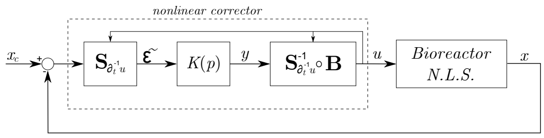

Fig. 2. Controller on the nonlinear system.

We give in fig. 3 an example of control of the biomass for two different profiles of biomass. The input and parameters are: $a_1 = 14.3$, $a_2 = 6.25$, $p_i = 100$ g.l^{-1}, $K_s = 0.5$ g.l^{-1}, $s_i = 50$ g.l^{-1}, $\mu = \mu_{\max}\frac{s}{k_s+s}\left(1 - \frac{p}{p_i}\right)$, $\mu_{\max} = 0.54$. We can see that the biomass follows the reference.

Moreover, as previously said, the parametrizing has been made independently of the specific growth rate $\mu(X)$; consequently, the control strategy is still valid for any expression of the growth rate. As an illustration, we give in fig. 4 an example of biomass control obtained with the specific growth rate of Monod (Monod [1942]), given by:

$$\mu(X) = \mu_{\max}\frac{s}{k_s + s}.$$

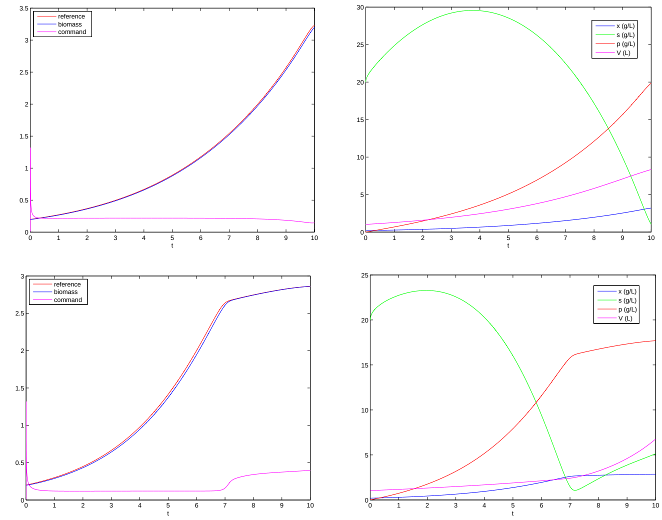

Fig. 3. Control of the biomass for two different profiles.

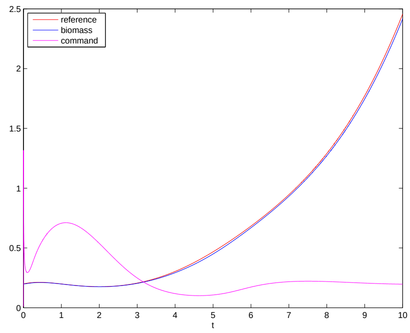

Fig. 4. Control of the biomass with a growth rate of Monod type.

Substrate and product control As the structure of equations governing the substrate s and the product p are similar to the biomass one, we can control those quantities in the same way than the biomass. Indeed, by denoting $\widetilde{S} := \widetilde{s} - s_i$, we have the equations:

$$\begin{cases} \partial_\tau \widetilde{S} = -\widetilde{S} - a_1 y_1 \\ \widetilde{S}(0) = s_0 - s_i \end{cases} \text{ and } \begin{cases} \partial_\tau \widetilde{p} = -\widetilde{p} + a_2 y_1 \\ \widetilde{p}(0) = p_0, \end{cases}$$

which have the same dynamic than (24); consequently, thay can be controlled with the respective correctors $-a_1 K(p)$ and $a_2 K(p)$, where $K(p)$ is the controller defined by (25).

We give on fig. 5 and exemple of control of substrate and product.

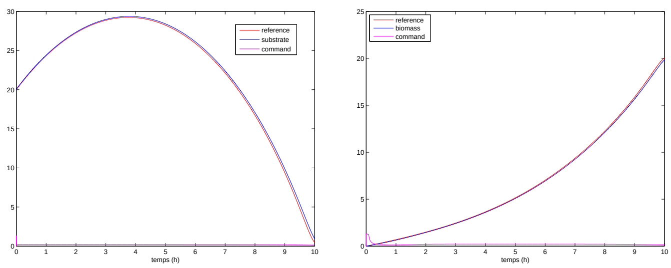

Fig. 5. Substrate (left) and Product (right) control.

Robustness to measurement noise We focus in this paragraph on the robustness of the control strategy to measurement noise. Then, we add to the measure a colored noised, obtained by filtering a blank noise of parameter σ with a first order filter of transfert function:

$$F(p) = \frac{a_m}{p + a_m}$$

Similarly, a filter of the form $\frac{a_u}{p+a_u}$ can be used if the computed control is too rough. We give in fig. 6 the results obtained with noisy measurement: the control is still good in spite of the noise.

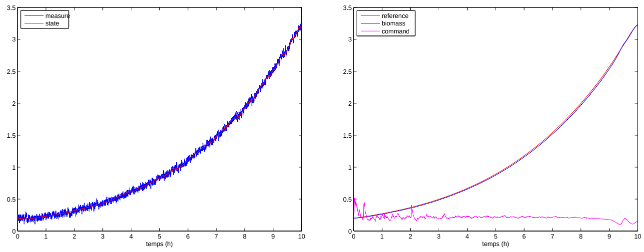

Fig. 6. Control of the biomass with noise parameters $\sigma = 0.1$ and $a_m = 200$, and with a command filter of parameter $a_u = 20$.

Perturbations robustness and compensation The proposed controler is robust to perturbations, as shown on fig. 7:

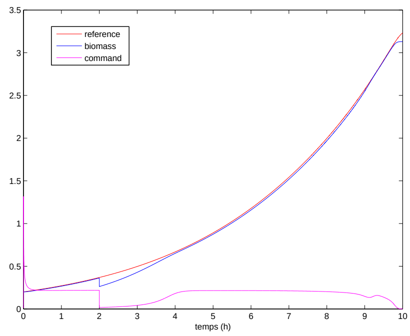

Fig. 7. Biomass control in presence of a perturbation $\Delta x = -0.1$ at $t_p = 2h$.

Remark 15. 1n error on initial condition can be seen as a perturbation at $t = 0$; so, the controller is also robust to initial condition errors.

The control strategy can be enhenced by adding a perturbation compensation in the parametrizing of the model. Indeed, we can see on fig. 8 that the reconstitution $\mathbf{S}^{-1}_{\partial_t^{-1} u} \circ \mathbf{C}(y)$ differs from the state X after the pertrubation. We can express this difference by considering a perturbation at time t_p, modeled by a term $\alpha \delta_{t_p}$, $\alpha > 0$, in the biomass[4] equation (the other equations are unchanged):

$$\begin{cases} \partial_t x = \mu(X)\, x - u\, x + \alpha \delta_{t_p} \\ \qquad \dots \end{cases}$$

[4] The same reasoning can of course be made for substrate or product perturbations.

Then, using the TST $\mathbf{S}_{\partial_t^{-1}u}$ and noting that $\mathbf{S}_{\partial_t^{-1}u}\left(\alpha\delta_{t_p}\right) = \alpha u(t_p)\,\delta_{\tau_p}$, we get from simple computations the following parametrizing of the model after the perturbation:

$$\begin{aligned}\widetilde{x} &= (\partial_\tau + 1)^{-1}(y_1) + y_2\,\mathrm{e}^{-(\cdot)} + u(t_p)\alpha\;(\partial_\tau + 1)^{-1}\left(\frac{1}{\widetilde{u}}\,\delta_{\tau_p}\right) \\ &= \mathbf{C}_1(y) + \alpha \mathrm{e}^{\tau_0 - \cdot}\;Y\left(\cdot - \tau_p\right), \qquad (26)\end{aligned}$$

where Y is the Heaviside function (the other equations are similar to (22)). Thus, a perturbation can analytically be taken into account by adding the term $\alpha \mathrm{e}^{\tau_0-\tau}\;Y\left(\tau - \tau_p\right)$ into the parametrization of $\widetilde{x}$. In practice, the perturbation is detected by a brutal variation of $\widetilde{x} - \mathbf{C}_1(y)$.

We give on fig. 8 an example of compensation of perturbation and its benefits on the parametrizing operator $\mathbf{C}(y)$ accuracy.

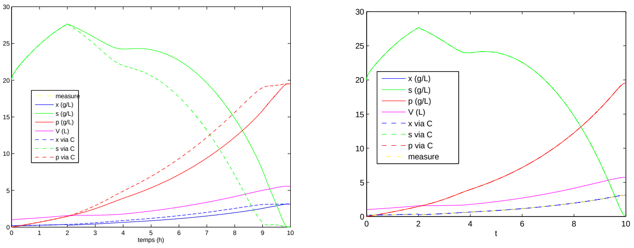

Fig. 8. Compensation of biomass perturbation.

6. CONCLUSION

The present paper must be seen as a brief introduction to a methodology based on operatorial transformations and devoted to nonlinear dynamic problems. The time-scale transformations were introduced and their interest in terms of decoupling was stated. Operatorial parametrization and some extensions such as parametric output and parametric equations were then introduced.

It has been shown on fed-batch bioreactor models that, in spite of some apparent complexity, efficient solutions can be found by using those operatorial transformations.

The obtained formulations allow to envisage practical implementation or even extensions, for exemple to the following class of bioreactor models:

$$\begin{cases} \partial_t\, x = G(X) - f(x)\,k(u) \\ \partial_t\, s = -a_1 G(X) + g(s_i - s)\,k(u) \\ \partial_t\, p = a_2 G(X) - h(p)\,k(u) \\ X(0) = X_0, \end{cases}$$

which can be transformed in the same way by TST and parametrizing Montseny [2009].

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