Simplification of dynamic problems by time-scale transformation: application to the nonlinear control with input positive constraints

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Abstract: In this paper, we show how time-scale transformations (TST) can be used for the control of nonlinear sytems subject to positive constraints. Such transformations, which consist in a change of the time variable, enable to define a new time-scale denoted τ in which the control problem becomes unconstrained and is therefore simplified. Classical methods such as dynamic feedback linearizing control design can then be used, leading to control laws that naturally fullfill the input positive constraints. The proposed method is applied on two concrete examples and compared with an other approach.

Keywords: input positive constraints; input saturation; nonlinear control; time-scale transformation.

1. INTRODUCTION

Consider a nonlinear system of the form:

$$\begin{cases} \dfrac{dx}{dt} &= f(x,u) \\ y &= h(x) \end{cases} \tag{1}$$

where $\forall t \geqslant 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, with n, m, $p \in \mathbb{N}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ are two continuously differentiable functions.
Assume that the control input u is subject to L input positive constraints:

$$k_l(u(t)) \geqslant 0,\ l = 1 : L, \forall t > 0, \tag{2}$$

where $k_l : \mathbb{R}^m \to \mathbb{R}$, $l = 1 : L$ are continuously differentiable functions. In the sequel, we denote Ω_c the subspace of $\mathbb{R}^m$ defined by:

$$\Omega_c := \{u \in \mathbb{R}^m,\ \text{such that } k_l(u) \geqslant 0,\ \forall l = 1 : L\}. \tag{3}$$

In this paper, we consider the problem of finding a control input u which drives the output variable y to a given setpoint y^* while fullfilling the positive constraints (2).

The class of constraints (2) include the saturation constraints as we indeed have:

$$u_m \leqslant u(t) \leqslant u_M \Leftrightarrow \begin{cases} u(t) - u_m \geqslant 0 \\ u_M - u(t) \geqslant 0 \end{cases} \tag{4}$$

where u_m and u_M are the lower and upper bounds of the saturation. This particular case of positive constraints has been widely studied in the literature, in the case of both linear and nonlinear systems. There are generally two ways to deal with such problems: (1) either we first design a controller without taking the constraints into account and then we try to compensate the effect of the saturation (this is the case of the anti-windup techniques (Tarbouriech and Turner, 2009)); (2) or we include the saturation from the beginning of the design process as it is the case with set invariance control design Blanchini (1999) or techniques based on the polytopic representation of the saturation (Cao and Lin, 2003; Bezzaoucha et al., 2013).

In (Doyle III, 1999), the author proposes to combine two techniques: the feedback linearizing control for nonlinear systems (Henson and Seborg, 1997), and an anti-windup technique developed by Zheng et al. (1994), which has the advantage to handle the case of saturations with time-variable bounds. An other interesting approach is the one proposed in Antonelli and Astolfi (2003) which integrates the bounds of the saturation directly in the control law. By forcing the control input equation to be of the form $\frac{du}{dt} = (u_M - u)(u - u_m)\gamma(x,u)$, it makes the set $[u_m, u_M]$ invariant; thus the input variable naturally fullfills the saturation constraint without degrading the performances of the closed-loop system. The drawback of this technique is that it relies on a Lyapunov function which is not always easy to find in the case of nonlinear systems. It also only handles saturation constraint with constant bounds. In the present paper, we propose to adapt the approach of Antonelli and Astolfi (2003) to more general positive constraints of the form (2). The approach no more relies on Lyapunov function but on Time-Scale Transformations (TST), that is on a nonlocal change of the time-variable. These transformations have successfully been applied on various problems, such as model singularities suppression and operatorial parametrization for nonlinear bioreactor control (Montseny, 2011, 2009). In this paper, we apply such a transformation to simplify the above mentionned constrained control problem into an unconstrained problem. Thus, classical control design methods can be used, leading to control laws that naturally fullfill the input positive constraints.

The paper is organized as follows. In section 2, definitions and results about time-scale transformations are given. Then, we show in section 3 how to use these time-scale transformation to simplify the problem of control of nonlinear systems subject to input positive constraints. Finally, some examples are given in section 4; the control strategies described in section 3 (combined with dynamic feedback linearizing control) are applied on concrete systems and compared with the method of Doyle III (1999).

2. TIME SCALE TRANSFORMATIONS

In this section, we give some definitions and results about time-scale transformations, that will be usefull in the sequel. More information about such transformations can be found in Montseny (2011, 2009).

2.1 Definitions and properties

Let $\mathbf{X}$ and $\mathbf{U}$ be some Banach spaces and $\mathfrak{X}$ a suitable space of trajectories with values in $\mathbf{X}$. In the sequel, ∂_t and ∂_t^{-1} denote respectively the time-derivation and time-integration operators, while the symbol $(\cdot)'$ represents the generic differentiation operation.

Definition 1. The "time-scale transformation" (TST) $\mathbf{S}_\varphi$ is the trajectorial transformation defined by:

$$\mathbf{S}_\varphi : x \mapsto x \circ \varphi^{-1} \tag{5}$$

where the strictly increasing (and thus invertible) function φ defines a new time-scale τ:

$$\tau := \varphi(t). \tag{6}$$

For convenience, we denote $\widetilde{x}$ the trajectory x transformed by the TST $\mathbf{S}_\varphi$:

$$\widetilde{x} := \mathbf{S}_\varphi(x) = x \circ \varphi^{-1}, \tag{7}$$

that is: $\widetilde{x}$ is the trajectory x expressed in time τ.

All time-scale transformations are invertible, which is essential to keep equivalence between models; the inverse of $\mathbf{S}_\varphi$ is simply given by:

$$\mathbf{S}_\varphi^{-1} = \mathbf{S}_{\varphi^{-1}}. \tag{8}$$

Definition 2. A time-scale-transformation is said *dynamic* if the clock φ is the result of a dynamic transformation of a function v, that is $\varphi = \boldsymbol{\varphi}(v)$ with $\boldsymbol{\varphi}$ a causal operator defined on a manifold $\mathcal{V}$ (of trajectories), such that $\forall v \in \mathcal{V}$, $\boldsymbol{\varphi}(v)$ is continuous and strictly increasing.
We denote $\mathbf{S}_{\boldsymbol{\varphi}}$ the operatorial function:

$$\mathbf{S}_{\boldsymbol{\varphi}} : v \mapsto \mathbf{S}_{\boldsymbol{\varphi}(v)}. \tag{9}$$

Remark 3. A dynamic time-scale transformation can be applied on the trajectory v itself :

$$\widetilde{v} = \mathbf{S}_{\boldsymbol{\varphi}(v)}\,(v) = v \circ \boldsymbol{\varphi}(v)^{-1}. \tag{10}$$

Note that because the operator $\boldsymbol{\varphi}$ is causal, this expression remains compatible with real time applications.

An important example of causal dynamic TST operator is given by $\boldsymbol{\varphi} = \partial_t^{-1}$. In particular, we have the following proposition.

Proposition 4. Let g a continuous and strictly positive function (i.e. $g \in C^0(I, \mathbb{R}^{+*})$) and x differentiable. Then:

$$\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}(g\,x') = [\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}(x)]'. \tag{11}$$

Roughly speaking:

$$\mathbf{S}_{\partial_t^{-1}\frac{1}{g}} :\; g\,\partial_t \mapsto \partial_\tau, \tag{12}$$

that is: by applying $\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}$, operator $g\partial_t$ is transformed into ∂_τ. Note that g can be any function of time, for example of the form $G(t, x, u)$.

Moreover, by denoting $\varphi = \partial_t^{-1}\frac{1}{g}$, and because g is strictly positive, we have:

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \int_0^t \frac{1}{g(s)} ds = \infty, \tag{13}$$

which gives $\varphi(\mathbb{R}^+) = \mathbb{R}^+$ and $\varphi^{-1}(\mathbb{R}^+) = \mathbb{R}^+$.

2.2 Transformation of differential models

Consider the generic dynamical system of the form:

$$g\,\partial_t x = F(x, u), \tag{14}$$

where g is a continuous and strictly positive function, and $F : \mathbf{X} \times \mathbf{U} \mapsto \mathbf{U}$ a continuously differentiable function.

The following proposition shows how, with a suitable TST, we can transform the differential model (14) into a simpler one.

Proposition 5. By the time-scale transformation $\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}$, equation (14) is transformed into:

$$\partial_\tau \widetilde{x} = F(\tilde{x}, \tilde{u}), \tag{15}$$

the correspondence between τ and t being indifferently defined by $\partial_t \tau = \frac{1}{g}$ or $\partial_\tau t = \widetilde{g}$.

With such a transformation, we can suppress some undesirable terms of a model by "absorbing" them into the new time derivative operator ∂_τ. Then, the resolution of some dynamic problems on such models can be simplified.

2.3 Lyapunov stability

Let g be a continuous and strictly positive function and consider a model of the form:

$$g\partial_t x = F(x),\; x(0) = x_0; \tag{16}$$

to which we apply the time-scale transformation $\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}$. From proposition 5, we then get:

$$\partial_\tau \tilde{x} = F(\tilde{x}),\; \tilde{x}(0) = x_0; \tag{17}$$

the correspondence between t and τ being equivalently defined by $\partial_\tau t = \tilde{g}$ or $\partial_t \tau = \frac{1}{g}$ with $\tau = \left(\partial_t^{-1}\frac{1}{g}\right)(t) =: \varphi(t)$ (which implies that $\tau(0) = 0$).
The stability of the two systems (16) and (17) are linked; we indeed have the following result.

Proposition 6. If $x^* \in \mathbf{X}$ is a (globally asymptotically) stable equilibrium point of (17), then x^* is also a (globally asymptotically) stable equilibrium point of (16).

Proof. Let x^* be a stable equilibrium point of (17); we have $F(x^*) = 0$. Given $\varepsilon > 0$, $\exists \delta > 0$ such that:

$$||x_0 - x^*||_{\mathbf{X}} \leqslant \delta \Rightarrow ||\tilde{x}(\tau) - x^*||_{\mathbf{X}} \leqslant \varepsilon,\; \forall \tau \geqslant 0, \tag{18}$$

where $\tilde{x}$ is the solution of (17). As $\varphi^{-1}(\mathbb{R}^+) = \mathbb{R}^+$ and $\forall \tau \geqslant 0$, $\tilde{x}(\tau) = x(\varphi^{-1}(\tau))$ with x the solution of (16):

$$||\tilde{x}(\tau) - x^*||_{\mathbf{X}} \leqslant \varepsilon, \forall \tau \geqslant 0 \Leftrightarrow ||x(t) - x^*||_{\mathbf{X}} \leqslant \varepsilon, \forall t \geqslant 0. \quad (19)$$

Therefore, x^* is also a stable equilibrium point of (16). Moreover, we deduce from (13) that:

$$\begin{aligned}\lim_{t\to\infty} ||x(t) - x^*||_{\mathbf{X}} &= \lim_{t\to\infty} ||\tilde{x}(\varphi(t)) - x^*||_{\mathbf{X}} \\ &= \lim_{\tau\to\infty} ||\tilde{x}(\tau) - x^*||_{\mathbf{X}} \qquad (20)\end{aligned}$$

So if x^* is asymptotically (respectively globally asymptotically) stable for system (17), it is also for (16).

3. CONTROL STRATEGIES

In this section, we show how TST transformations can simplify a problem of control of a nonlinear system subject to input positive saturation. The particular case of SISO systems with saturation constraints is first presented; then the general result is given for MIMO systems.

3.1 SISO systems with saturation constraints

The method takes its inspiration from the work of Antonelli and Astolfi (2003). The idea is rather simple: to ensure that the input u will fullfill the saturation constraint (4), we search a dynamic control law of the form:

$$\frac{du}{dt} = (u_M - u)(u - u_m)\gamma(x, u), \quad (21)$$

with $\gamma : \mathbb{R}^n \times]u_m, u_M[\mapsto \mathbb{R}$. The closed-loop system will therefore be written:

$$\begin{cases} \dfrac{dx}{dt} &= f(x, u) \\ \dfrac{du}{dt} &= (u_M - u)(u - u_m)\gamma(x, u) \\ y &= h(x) \end{cases} \quad (22)$$

By application of the time-scale transformation

$$\mathbf{S}_{\partial_t^{-1}((u_M - u)(u - u_m))}, \quad (23)$$

we then get a new closed-loop system (see proposition 5):

$$\begin{cases} \dfrac{d\tilde{x}}{d\tau} &= \dfrac{f(\tilde{x}, \tilde{u})}{(u_M - \tilde{u})(\tilde{u} - u_m)} =: \overline{f}(\tilde{x}, \tilde{u}) \\ \dfrac{d\tilde{u}}{d\tau} &= \gamma(\tilde{x}, \tilde{u}) \\ \tilde{y} &= h(\tilde{x}) \end{cases} \quad (24)$$

defined in a new time-scale τ. The bijective correspondence between the times t and τ is given by:

$$\partial_t \tau = (u_M - u)(u - u_m) \text{ and } \partial_\tau t = \frac{1}{(u_M - \tilde{u})(\tilde{u} - u_m)}. \quad (25)$$

Thus, the TST (23) transformed the initial problem (22) into the unconstrained problem (24) on which classical nonlinear control design methods can be applied.

Proposition 7. Consider the system (1) and a point $x^* \in \mathbb{R}^n$. Suppose that there exists a bounded function $\gamma : \mathbb{R}^n \times]u_m, u_M[\mapsto \mathbb{R}$ such that:

(*i*) x^* is a stable equilibrium point of:

$$\frac{d\tilde{x}}{d\tau} = \frac{f(\tilde{x}, \tilde{u})}{(u_M - \tilde{u})(\tilde{u} - u_m)}; \; \tilde{x}(0) = x_0 \in \mathbb{R}^n, \quad (26)$$

with $\tilde{u}$ solution of:

$$\frac{d\tilde{u}}{d\tau} = \gamma(\tilde{x}, \tilde{u}); \; \tilde{u}(0) = u_0 \in]u_m, u_M[. \quad (27)$$

(*ii*) the trajectories $(\tilde{x}, \tilde{u})$ solution of system (26,27) are bounded.

Then the control law defined by:

$$\frac{du}{dt} = (u_M - u)(u - u_m)\gamma(x, u); \; u(0) = u_0, \quad (28)$$

stabilizes the system (1) at x^*, while ensuring that the constraint (4) is fulfilled for all $t > 0$.

Proof. The stabilization of (1) at x^* directly results from proposition 6 in the particular case where $F : \tilde{x} \mapsto \frac{f(\tilde{x}, \tilde{u}(\tilde{x}))}{(u_M - \tilde{u}(\tilde{x}))(\tilde{u}(\tilde{x}) - u_m)}$, $\tilde{u}(\tilde{x})$ being the solution of (27). Moreover, by denoting $y_1 = u_M - u$ and $y_2 = u - u_m$ we can show that the system:

$$\frac{dy_1}{dt} = -y_1 y_2 \gamma(x, u_M - y_1); \; \frac{dy_2}{dt} = y_1 y_2 \gamma(x, y_2 + u_m),$$

is a positive system as we have: $\forall i \in \{1, 2\}, y_i = 0 \Rightarrow \frac{dy_i}{dt} = 0, \forall x \in \mathbb{R}^n$. Moreover, as $(y_1(t), y_2(t)) = (0, 0)$ is solution of these equations, it follows from Cauchy-Lipschitz theorem that $y_1(t) > 0$ and $y_2(t) > 0$ providing that $y_1(0)$ and $y_2(0) > 0$, that is $u_m < u(t) < u_M$ providing that $u_0 \in]u_m, u_M[$.

3.2 MIMO systems with general positive constraints

The previous result can easily be extended to the case of MIMO systems with L input positive constraints of the form (2). In that case, we have the following result:

Proposition 8. Consider the system (1) and a point $y^* \in \mathbb{R}^p$. Assume there exists a unique $x^* \in \mathbb{R}^n$ such that $h(x^*) = y^*$. Given L positive functions $K_l : \Omega_c \mapsto \mathbb{R}^+$, $l = 1 : L$ such that, $\forall l$, $k_l(u) = 0 \Leftrightarrow K_l(u) = 0$, let's denote τ the time variable defined by:

$$\partial_t \tau = \prod_{l=1}^{L} K_l(u) \text{ and } \partial_\tau t = \frac{1}{\prod_{l=1}^{L} K_l(u)}. \quad (29)$$

Suppose that there exists a dynamic control law $\tilde{u} = (\tilde{u}_1, \dots \tilde{u}_m)$ defined by:

$$\frac{d\tilde{u}_i}{d\tau} = \gamma_i(\tilde{x}, \tilde{u}), \; i = 1 : m, \quad (30)$$

with $\gamma_i : (\tilde{x}, \tilde{u}) \in \mathbb{R}^n \times \Omega_c \mapsto \gamma_i(\tilde{x}, \tilde{u})$, $i = 1 : m$ some bounded functions, such that:

(*i*) x^* is a globally asymptotically stable equilibrium point of:

$$\frac{d\tilde{x}}{d\tau} = \frac{f(\tilde{x}, \tilde{u})}{\prod_{l=1}^{L} K_l(\tilde{u})} \left(=: \overline{f}(\tilde{x}, \tilde{u})\right); \; \tilde{x}(0) = x_0 \in \mathbb{R}^n, \quad (31)$$

(*ii*) the trajectories $(\tilde{x}, \tilde{u})$ solution of system (31,30) are bounded.

Then the control law defined by:

$$\frac{du_i}{dt} = \left(\prod_{l=1}^{L} K_l(u)\right) \gamma_i(x, u), \; i = 1 : m, \quad (32)$$

stabilizes the system (1) at x^*, while ensuring that the constraints (2) are fulfilled for all $t > 0$.

Proof. The proof is similar to the one of proposition 7. The stabilization of (1) at x^* directly results from proposition 6 in the particular case where $F : \tilde{x} \mapsto$

$\frac{f(\tilde{x},\tilde{u}(\tilde{x}))}{\prod_{l=1}^{L} K_l(\tilde{u}(\tilde{x}))}$, $\tilde{u}(\tilde{x})$ being the solution of (30). Moreover, we have, $\forall l = 1 : L$:

$$\frac{d(k_l(u))}{dt} = \sum_{i=1}^{m} \frac{du_i}{dt}\frac{\partial k_l}{\partial u_i} = \left(\prod_{j=1}^{L} K_j(u)\right) \sum_{i=1}^{m} \gamma_i(x,u) \frac{\partial k_l}{\partial u_i}.$$

As $\forall l$, $k_l(u) = 0 \Leftrightarrow K_l(u) = 0$, it follows that $\forall l$, $k_l(u) = 0 \Rightarrow \frac{d(k_l(u))}{dt} = 0$, $\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. Moreover, as $k_l(u) = 0$, $l = 1 : L$ is solution of these equations, it follows from Cauchy-Lipschitz theorem that $k_l(u(t)) > 0$ providing that $k_l(u(0)) > 0$.

Remark 9. An obvious choice for the functions K_l is to take $K_l = k_l$.

4. EXAMPLES

4.1 Dynamic linearizing control

As said previously, the classical form of transformed model (24) allows to use several classical approaches for the design of the control law, i.e. for the choice of γ. We use in this paper dynamic feedback linearization (Henson and Seborg, 1997) and give here-after a few reminder.

As it will be the case in the examples, we will only consider some systems such that:

- $p = m$ (same number of inputs and outputs);
- $y = Cx$ with $C \in \mathbb{R}^{p\times n}$;
- the matrix:
$$A(\tilde{x},\tilde{u}) = C\nabla_{\tilde{u}}\overline{f} := C\left[\frac{\partial \overline{f}_i}{\partial \tilde{u}_j}\right]_{i,j=1:p} \quad (33)$$
is nonsingular for all $(\tilde{x},\tilde{u}) \in (X \times \Omega_c) \subset \mathbb{R}^n \times \mathbb{R}^m$.

In this particular case, the dynamic feedback linearizing controller is given by:

$$\frac{d\tilde{u}}{d\tau} = A^{-1}(\tilde{x},\tilde{u})\left[\tilde{v} - B(\tilde{x},\tilde{u})\overline{f}(\tilde{x},\tilde{u})\right] =: \gamma(\tilde{x},\tilde{u}), \quad (34)$$

where $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_p)^T \in \mathbb{R}^p$ and:

$$B(\tilde{x},\tilde{u}) = \nabla_{\tilde{x}}\overline{f} := \left[\frac{\partial \overline{f}_i}{\partial \tilde{x}_j}\right]_{i=1:p,j=1:n}. \quad (35)$$

The variable $\tilde{v}$ can be designed in order to impose a stable linear equation for the output y_j as we have (Henson and Seborg, 1997):

$$\frac{d^2\tilde{y}_j}{d\tau^2} = \tilde{v}_j. \quad (36)$$

A simple choice for $\tilde{v}$ is therefore:

$$\tilde{v} = -D_0(\tilde{y}-y^*) - D_1\frac{d\tilde{y}}{d\tau} = -D_0(C\tilde{x}-y^*) - D_1C\frac{d\tilde{x}}{d\tau} \quad (37)$$

with:

$$D_0 = \begin{bmatrix} \alpha_0^1 & & \\ & \ddots & \\ & & \alpha_0^p \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} \alpha_1^1 & & \\ & \ddots & \\ & & \alpha_1^p \end{bmatrix}. \quad (38)$$

For each i we thus obtain a rational transfer function between y_i and y_i^*, whose characteristic polynomial is given by $\alpha_0^i + \alpha_1^i s + s^2$.

4.2 A scalar nonlinear example

In (Doyle III, 1999), F.J.Doyle proposed a method combining the feedback linearizing technique with the anti-windup IMC of Zheng et al. (1994) (which is dedicated to linear systems) for the control of nonlinear systems with input saturations. In order to compare our TST-based control strategy with the one of F.J.Doyle, we will consider the same nonlinear system:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{e^x}{100}(2u - x) =: f(x,u) \\ y = x =: h(x) \end{cases} \quad (39)$$

whith u the control input which is subject to a saturation constraint of the form (4) with $u_m = -1$ and $u_M = 1$. It is a particular case of system (1) with $m = p = n = 1$.

Following the strategy proposed in section 3, we transform (39) with TST defined by (21), which leads to the global controlled system (24) with γ defined by (34-38) and:

$$\overline{f}(\tilde{x},\tilde{u}) = \frac{f(\tilde{x},\tilde{u})}{(u_M - \tilde{u})(\tilde{u} - u_m)}. \quad (40)$$

After some computations:

$$\frac{\partial\overline{f}}{\partial\tilde{x}} = \frac{\partial_1 f(\tilde{x},\tilde{u})}{(u_M - \tilde{u})(\tilde{u} - u_m)} \quad (41)$$
$$\frac{\partial\overline{f}}{\partial\tilde{u}} = \frac{\partial_2 f(\tilde{x},\tilde{u})(u_M-\tilde{u})(\tilde{u}-u_m) - (u_M+u_m-2\tilde{u})f(\tilde{x},\tilde{u})}{(u_M-\tilde{u})^2(\tilde{u}-u_m)^2}$$

where:

$$\partial_1 f(x,u) = -\frac{e^x}{100}\left[2u - x - 1\right]; \; \partial_2 f(x,u) = \frac{2e^x}{100}. \quad (42)$$

We use the following parameters values:

$$\alpha_1^1 = 2\xi\omega; \; \alpha_0^1 = \omega^2 \text{ with } \xi = 0.7; \; \omega = 6.10^{-1}. \quad (43)$$

The results are given in figure 1 in the case where $y^* = 1$. Several controllers have been considered for comparison. The input-output linearization (IOLin) controller is the one obtained by the feedback linearizing design:

$$w = \frac{5y^* - 5y + ye^y}{2e^y} \quad (44)$$

that leads (when $u = w$) to the closed loop system $\frac{dy}{dt} = \frac{y^*-y}{20}$. Both the unconstrained case ($u = w$) and saturated case ($u = \mathrm{sat}(w)$) were simulated. The anti-windup Internal Model Controller (AW-IMC) is the one proposed by Zheng et al. (1994). The anti-windup input-output linearization controller (AW-IOLin) is the one proposed by Doyle III (1999).

As we can notice, the TST controller behaves satisfactorily. It gives good results, the relative error $e = \left|\frac{y-y^*}{y^*}\right|$ being even smaller than the ones obtained with the other controllers. The TST controller also prevent the input from saturating, which is not the case of the other controllers.

4.3 A concrete MIMO system

Let us now consider the more concrete example of a Multi-Stage Continuous Fermentor (MSCF), which is a device composed of R reactors connected in series. Such a device is used for the study of the wine fermentation (Clement et al., 2011), which can be summarized by the 2 following main reactions: (1) the yeasts Y grow on the nitrogen N; (2) the sugar S is enzymatically degraded (by the yeast) into ethanol E and CO_2, and inhibited by the ethanol. The volumes ν_j, $j = 1\!:\!R$ of the reactors remain constant, which means that the output flow rate of each reactor of the MSCF is equal to its input flow rate. The first reactor is fed with the must which only contains sugar

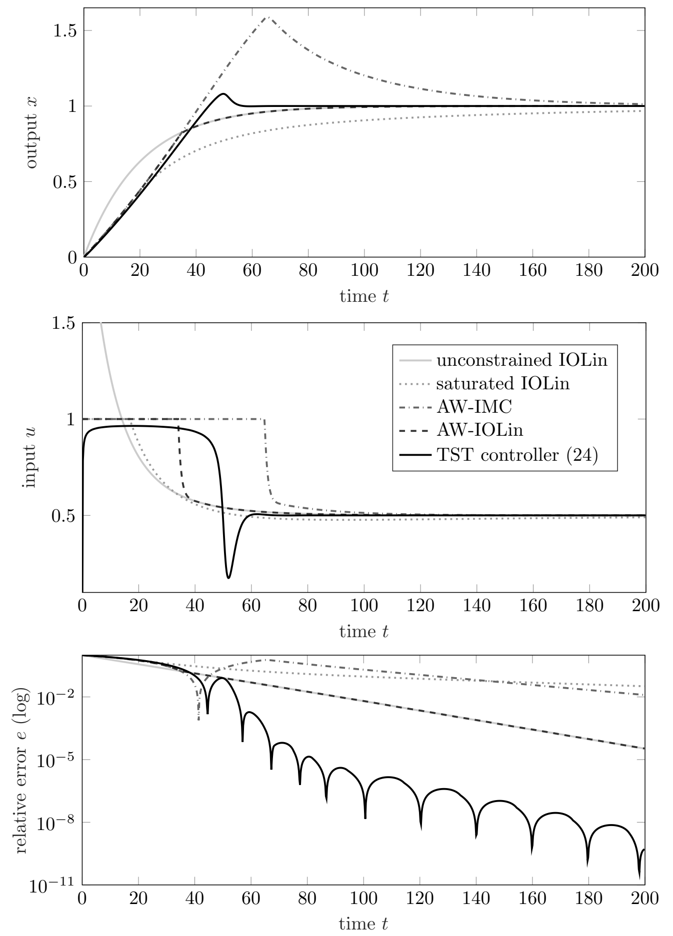

Fig. 1. **A scalar nonlinear example**: control of system (39). IOLin, AW and IMC stand for "Input-Output Linearization", "Anti-Windup" and "Internal Model Control".

(with concentration S_0) and nitrogen (with concentration N_0). Each of the $R-1$ other reactors are fed with a fraction of the output of the previous reactor without any external addition. The input flow rates Q_j, $j = 1{:}R$ of the reactors are controlled independently by some piston pumps, the only constraint (coming from the cascade structure of the device) being that the input flow rate Q_j of the j^{th} reactor has to be smaller than the output flow rate Q_{j-1} of the $(j-1)^{\text{th}}$ reactor:

$$Q_m < Q_R < Q_{R-1} < \ldots < Q_2 < Q_1 < Q_M, \tag{45}$$

with Q_m and Q_M the minimal and maximal flow rates which can be applied.

A model of such a system is given by, $\forall j = 1 : R$:

$$\begin{cases} \dfrac{dY_j}{dt} = \mu_1(N_j)Y_j + D_j(Y_{j-1} - Y_j) \\ \dfrac{dN_j}{dt} = -k_1\mu_1(N_j)Y_j + D_j(N_{j-1} - N_j) \\ \dfrac{dE_j}{dt} = \mu_2(E_j, S_j)Y_j + D_j(E_{j-1} - E_j) \\ \dfrac{dS_j}{dt} = \underbrace{-k_2\mu_2(E_j, S_j)Y_j + D_j(S_{j-1} - S_j)}_{=:f_j^S(Y_j, N_j, E_j, S_j, S_{j-1}, D_j)} \end{cases} \tag{46}$$

with:

$$\mu_1(N) = \frac{\mu_1^{max} N}{K_N + N}, \quad \mu_2(E, S) = \frac{\mu_2^{max} S}{K_S + S}\frac{K_E}{K_E + E}, \tag{47}$$

where Y_j, N_j, E_j and S_j are the biomass (yeast), nitrogen, ethanol and sugar concentrations in the j^{th} reactor, $Y_0(= 0)$, N_0, $E_0(= 0)$ and S_0 the biomass, nitrogen, ethanol and sugar concentrations of the must feeding the first reactor, $D_j = \frac{Q_j}{\nu_j}$ the dilution rate of the j^{th} reactor, k_1 and k_2 the yield coefficients, and μ_1^{max}, μ_2^{max}, K_N, K_E and K_S some positive constants. To complete the system, we add the following initial conditions: $\forall j = 1 : R$,

$$(Y_j(0), N_j(0), E_j(0), S_j(0)) = (Y_{init}, N_0, 0, S_0), \tag{48}$$

where Y_{init} is the initial concentration of the yeast when the inoculation is performed.

We consider the problem of the control of the sugar concentrations $y = (S_1, \ldots, S_R)^T$ with $u = Q = (Q_1, \ldots, Q_R)^T$ as control input in a MSCF with $R = 4$ reactors. The control input Q is subject to the constraint (45) that can be rewritten in a set of $L = R + 1$ positive constraints of the form (2) where:

$$k_l(Q) = Q_{l-1} - Q_l,\ l = 1 : R + 1, \tag{49}$$

with $Q_0 = Q_m$ and $Q_{R+1} = Q_M$. We denote $S^* = (S_1^*, \ldots, S_R^*)^T$ the setpoint values of sugar concentrations; In the sequel, we denote $x := (\xi_1, \xi_2, \xi_3, \xi_4)^T$ with $\xi_i = (Y_i, E_i, N_i, S_i)$, $i = 1 : 4$.

This problem has already been considered in Casenave et al. (2014) where the method of Doyle has been applied. In the present paper, we propose to test the TST controller (32) with γ defined by (34-38) and $\overline{f}$ defined by (31). To implement the controller, we need to compute the following quantities that appear in the expression of γ:

$$\nabla_{\tilde{\xi}_j}\overline{f}_i^S = \frac{\nabla_{\tilde{\xi}_j} f_i^S}{\prod_l K_l(Q)} \text{ where } \overline{f}_i^S(\tilde{x}, \tilde{Q}) = \frac{f_i^S(\tilde{x}, \tilde{Q})}{\prod_l K_l(\tilde{Q})} \tag{50}$$

$$\text{with: } \frac{\partial f_i^S}{\partial \tilde{N}_j} = 0;\ \frac{\partial f_i^S}{\partial \tilde{Y}_j} = \begin{cases} -k_2\mu_2(\tilde{E}_i, \tilde{S}_i) \text{ if } j = i \\ 0 \text{ else} \end{cases} \tag{51}$$

$$\frac{\partial f_i^S}{\partial \tilde{E}_j} = \begin{cases} -k_2\partial_1\mu_2(\tilde{E}_i, \tilde{S}_i) \text{ if } j = i \\ 0 \text{ else} \end{cases} \tag{52}$$

$$\frac{\partial f_i^S}{\partial \tilde{S}_j} = \begin{cases} -k_2\partial_2\mu_2(\tilde{E}_i, \tilde{S}_i) - \tilde{D}_i \text{ if } j = i \\ \tilde{D}_i \text{ if } j = i - 1 \text{ and } i \neq 1 \\ 0 \text{ else} \end{cases} \tag{53}$$

$$\frac{\partial \overline{f}_i^S}{\partial \tilde{Q}_j} = \frac{-f_i^S(\tilde{x}, \tilde{Q})}{(\prod_l K_l(\tilde{Q}))^2}\frac{\partial(\prod_l K_l(\tilde{Q}))}{\partial \tilde{Q}_j} + \frac{1}{\prod_l K_l(\tilde{Q})}\begin{cases} \dfrac{\tilde{S}_{i-1} - S_i}{\nu_i} \text{ if } j = i \\ 0 \text{ else} \end{cases} \tag{54}$$

Two controllers corresponding to two different sets of functions K_l, $l = 1 : R + 1$ have been considered:

- TST controller 1: $K_l(Q) = Q_{l-1} - Q_l (= k_l(Q))$;
- TST controller 2: $K_l(Q) = \frac{Q_{l-1} - Q_l}{K_Q + Q_{l-1} - Q_l}$, $K_Q > 0$.

We thus have:

$$\frac{\partial(\prod_l K_l(Q))}{\partial Q_j} = \prod_{l, l\neq j, l\neq j+1} K_l(\tilde{Q}) \tag{55}$$

$$\times \begin{cases} (Q_{j+1} + Q_{j-1} - 2Q_j) & \text{(TST controller 1)} \\ \frac{K_Q(Q_{j+1} + Q_{j-1} - 2Q_j)(Q_{j-1} - Q_{j-1} + K_Q)}{(Q_j - Q_{j+1} + K_Q)^2(Q_{j-1} - Q_j + K_Q)^2} & \text{(TST controller 2)} \end{cases}$$

The values of the model parameters used for the numerical simulations are given here after:

$k_1 = 0.0606\,[-]$	$\mu_1^{max} = 1.34\,[h^{-1}]$	$K_N = 1.57\,[g.L^{-1}]$
$k_2 = 2.17\,[-]$	$\mu_2^{max} = 1.45\,[h^{-1}]$	$K_E = 14.1\,[g.L^{-1}]$
$N_0 = 0.465\,[g.L^{-1}]$	$S_0 = 200\,[g.L^{-1}]$	$K_S = 0.0154\,[g.L^{-1}]$
$Y_{init} = 0.04\,[g.L^{-1}]$	$Q_m = 0\,[L.h^{-1}]$	$Q_M = 0.3\,[L.h^{-1}]$

To initialize the system, we simulated the system in open-loop from $t = 0$ to $t = 300$, with a constant control input $Q_0 = (0.17, 0.133, 0.09, 0.084)\,[L.h^{-1}]$; at time $t = 300$, the system was therefore at equilibrium. From $t = 300$ to $t = 320$, we then simulated the closed-loop system with the TST controllers 1 and 2 and the following parameters values: $\alpha_1^i = 2\xi\omega$ and $\alpha_0^i = \omega^2$, with $\xi = 0.7$ and $\omega = 6.10^6$ for the TST controller 1 and $\xi = 0.7$, $\omega = 6.10^{-1}$ and $K_Q = 0.001$ for the TST controller 2. The setpoint values were taken equal to $S^* = (184, 160, 140, 50)\,[g.L^{-1}]$ and the volumes of the $R = 4$ reactors were $\nu = (1, 0.7, 0.5, 0.7)\,[L]$. The results are given in figure 2.

We can see that both controllers manage to drive the output y to the setpoint value, but that the transitory behavior of the control input u is different. The TST controller 2 leads to smoother dynamic, which can be explained by the form of the associated functions K_l. Indeed, in comparison with the TST controller 1, the functions K_l of the TST controller 2 only influence the dynamic of u when Q_l is close to Q_{l-1}; when $Q_{l-1} - Q_l$ goes to infinity, $K_l(Q)$ goes to 1.

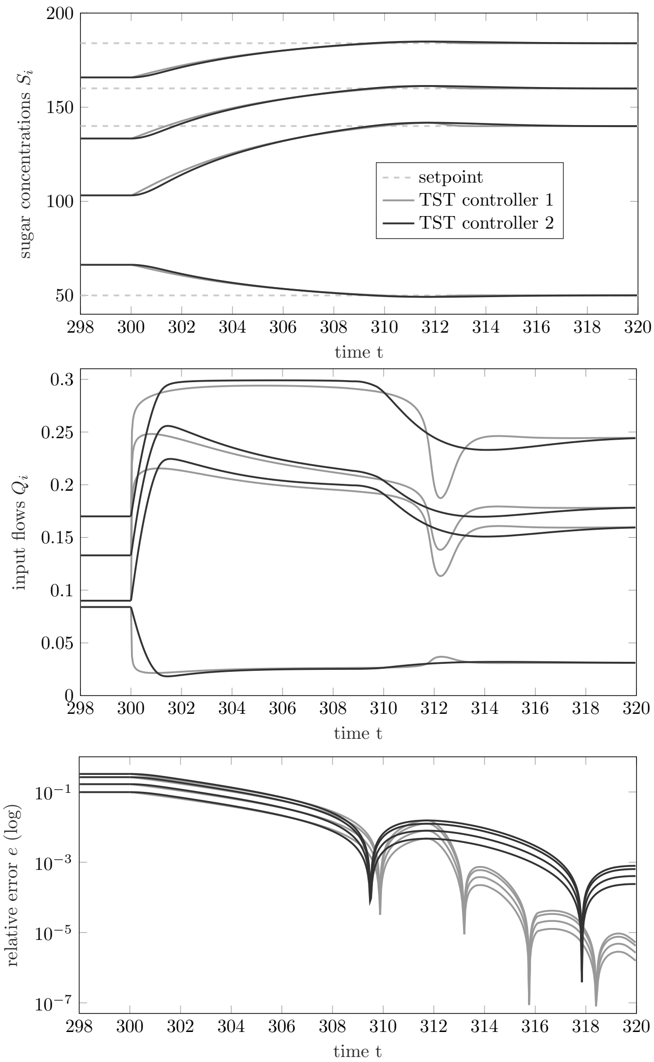

Fig. 2. **A concrete MIMO example**: control of the system (46), representing a Multi-Stage Continuous Fermenter (MSCF) with $R = 4$ reactors.

5. CONCLUSION

In this paper, we propose some strategies for the control of nonlinear systems subject to positive input constraints. These strategies are based on Time-Scale Transformations (TST) from which we define a new time-scale τ. When considered in this new time-scale, the control problem is no more constrained and classical control design can be performed. This approach has been tested on several examples and gives promising results. However many questions remain unanswered. In particular the convergence speed of the closed loop system is only controlled in the time τ; the impact on the convergence speed in the original time t is currently under study.

REFERENCES

Antonelli, R. and Astolfi, A. (2003). Continuous stirred tank reactors: easy to stabilise? *Automatica*, 39(10), 1817–1827.

Bezzaoucha, S., Marx, B., Maquin, D., and Ragot, J. (2013). Stabilization of nonlinear systems subject to actuator saturation. In *Fuzzy Systems (FUZZ), 2013 IEEE International Conference on*, 1–6. IEEE.

Blanchini, F. (1999). Set invariance in control. *Automatica*, 35, 1747–1767.

Cao, Y.Y. and Lin, Z. (2003). Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation. *IEEE Transactions on Fuzzy Systems*, 11(1), 57–67.

Casenave, C., Dochain, D., Harmand, J., Perez, M., Rapaport, A., and Sablayrolles, J.M. (2014). Control of a Multi-Stage Continuous Fermentor for the study of the wine fermentation. In *IFAC WC 2014*. Cap Town (South Africa).

Clement, T., Perez, M., Mouret, J., Sablayrolles, J., and Camarasa, C. (2011). Use of a continuous multistage bioreactor to mimic winemaking fermentation. *International Journal of Food Microbiology*, 150(1), 42–49.

Doyle III, F.J. (1999). An anti-windup input-output linearization scheme for SISO systems. *Journal of Process Control*, 9(3), 213–220.

Henson, M.A. and Seborg, D.E. (1997). *Nonlinear process control*, chapter Feedback linearizing control, 149–231.

Montseny, E. (2009). Operatorial transformations of dynamic problems and applications. *PhD Thesis*.

Montseny, E. (2011). Control of fed-batch bioreactors models by means of dynamic time-scale transformation and operatorial parametrization. In *IFAC WC 2011*, volume 44, 8439–8444.

Tarbouriech, S. and Turner, M. (2009). Anti-windup design: an overview of some recent advances and open problems. *IET control theory & applications*, 3(1), 1–19.

Zheng, A., Kothare, M.V., and Morari, M. (1994). Anti-windup design for internal model control. *International Journal of Control*, 60(5), 1015–1024.