

Dissipative terms and local time-stepping improvements in a spatial high order Discontinuous Galerkin scheme for the time-domain Maxwell's equations

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Abstract

In this paper, we present some improvements, in terms of accuracy and speed-up, for a particular well adapted Discontinuous Galerkin method devoted to the time-domain Maxwell equations. First, to reduce spurious modes on very distorted meshes, the addition of dissipative terms as penalization in the numerical scheme is studied and compared on examples. Second, in order to increase the efficiency of the method, a multi-class local time-stepping strategy is presented and its validation and advantages are highlighted on different examples.

Key words: Maxwell's equations, Discontinuous Galerkin method, local time-stepping

1 Introduction

In order to limit dispersive and dissipative errors [1] generated by classical schemes used to solve the time-domain Maxwell equations like FDTD [2] [3] or FVTD [4],[5],[6], some other methods as Discontinuous Galerkin (DG) methods [7], using high order spatial approximation of the fields in each cell, have been studied. Such methods are generally used with unstructured meshes and naturally allow spatial refinements when necessary, as for example near walls or in presence of materials with high dielectric contrasts. We have developed for the Maxwell equations a DG method based on a leap-frog scheme in time

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and a non-dissipative fluxes formulation [8]. However, the use of unstructured meshes can imply the presence of very distorted and small cells. Consequently, to ensure accuracy and stability, an increase of the spatial order and small time steps are required in the current available scheme. In this paper, after a first section in which we recall the principles and advantages of the particular DG method under consideration, we study in the second section the way to reduce spurious modes, and then the way to increase accuracy by introducing a penalization of the fluxes by dissipative terms. In particular, via a mathematical analysis, we show the convergence of the scheme even when using an order 1 spatial approximation, which is not the case without those dissipative terms [10]. Finally, in a third section, we develop a multi-class local time-stepping strategy adapted to the leap-frog time scheme classically used for Maxwell's equations [2]. The efficiency of this approach is compared with various local time-stepping strategies in terms of reduction of CPU time. Some validations on examples are given, in particular the possibility of using this local time-stepping method on cavity problems is emphasized.

2 Principles and advantages of the Discontinuous Galerkin method considered

Let Ω be a bounded open subset of \mathbb{R}^3 whose boundary is $\partial\Omega$, and let n denotes the unit outward normal to Ω . Let $\varepsilon(x)$, $\mu(x)$ and $\sigma(x)$ denote, respectively, the permittivity, the permeability and the conductivity of the medium.

We consider the problem described by the Maxwell equations:

Find $(E, H) : \Omega \times]0, T[\rightarrow \mathbb{R}^3 \times \mathbb{R}^3$ such that:

$$\begin{cases} \varepsilon \partial_t E + \sigma E = \nabla \times H, \\ \mu \partial_t H = -\nabla \times E, \\ n \times E = 0 \text{ on } \partial\Omega. \end{cases} \quad (1)$$

where E and H define the electric and magnetic fields. The boundary condition is not restrictive because it is used both to treat closed problems as cavity and to bound the Perfectly Matched Layers (PML) domain [11]. Consider a set \mathcal{T} of hexahedral elements $(K_i)_{i=1..N}$ being a partition of Ω . We introduce the following approximate space:

$$\mathbf{V}_r = \left\{ \mathbf{v} \in [L^2(\Omega)]^3 ; \forall K \in \mathcal{T}, DF_K^* \mathbf{v}|_K \circ F_K \in [Q_r(\hat{K})]^3 \right\}, \quad (2)$$

where $\hat{K} = [0, 1]^3$ is the unit cube, $\forall K \in \mathcal{T}$, $F_K : \hat{K} \rightarrow K$ denotes the trilinear mapping which associates the vertices of each element, $Q_r(\hat{K})$ is the space of polynomials of degree at most equal to $r \in \mathbb{N}^*$ in each variable on \hat{K} and

DF_K and J_K are respectively the Jacobian matrix and its determinant associated with the map F_K . Moreover, to each $K \in \mathcal{T}$, we associate the outward unit normal n_K .

Usually, for DG methods, E and H fields are approximated by polynomials on each cell. In our case, we approximate by polynomials the fields $DF_K^* E \circ F_K$ and $DF_K^* H \circ F_K$ on \hat{K} . It is not a strange choice since the Jacobian matrix is the essential ingredient to build a conform H-curl approximation [12]. As we shall see it later, this will imply interesting properties for memory storage.

Finally, we consider the following semi-discrete DG method:

find $(E_h(\cdot, t), H_h(\cdot, t)) \in \mathbf{V}_r \times \mathbf{V}_r$ such that : $\forall K \in \mathcal{T}$ and $\forall \psi, \phi \in \mathbf{V}_r$

$$\begin{cases} \int_K \varepsilon \partial_t E_h \cdot \psi \, dx + \int_K \sigma E_h \cdot \psi \, dx = \int_K \nabla \times H_h \cdot \psi \, dx \\ + \int_{\partial K} \left(\alpha_{\partial K}^K \llbracket n_K \times (E_h \times n_K) \rrbracket_{\partial K}^K + \beta_{\partial K}^K \llbracket H_h \times n_K \rrbracket_{\partial K}^K \right) \cdot \psi \, ds, \\ \int_K \mu \partial_t H_h \cdot \phi \, dx = - \int_K \nabla \times E_h \cdot \phi \, dx \\ + \int_{\partial K} \left(\gamma_{\partial K}^K \llbracket E_h \times n_K \rrbracket_{\partial K}^K + \delta_{\partial K}^K \llbracket n_K \times (H_h \times n_K) \rrbracket_{\partial K}^K \right) \cdot \phi \, ds, \end{cases} \quad (3)$$

where $\alpha_{\partial K}^K, \beta_{\partial K}^K, \gamma_{\partial K}^K, \delta_{\partial K}^K$ are parameters constant per face and $\llbracket v \rrbracket_{\Gamma}^K = (v|_{K'})_{\Gamma} - (v|_K)_{\Gamma}$ the jump across the boundary $\Gamma = K' \cap K$. When Γ is a boundary face (ie $\Gamma = K \cap \partial\Omega$) then K' does not exist and we simply define $\llbracket v \rrbracket_{\Gamma}^K = -(v|_K)_{\Gamma}$.

The coefficients $\alpha_{\partial K}^K, \beta_{\partial K}^K, \gamma_{\partial K}^K, \delta_{\partial K}^K$ are chosen such that (1) and (3) are equivalent problems (in the continuous sense) and to ensure a conservative formulation:

- $\forall \Gamma = K \cap K', \alpha_{\Gamma}^K = \alpha_{\Gamma}^{K'} = 0, \delta_{\Gamma}^K = \delta_{\Gamma}^{K'} = 0, \beta_{\Gamma}^K = \beta_{\Gamma}^{K'} = -\frac{1}{2}, \gamma_{\Gamma}^K = \gamma_{\Gamma}^{K'} = \frac{1}{2}$
- $\forall \Gamma = K \cap \partial\Omega, \beta_{\Gamma}^K = 0 = \delta_{\Gamma}^K = \alpha_{\Gamma}^K, \gamma_{\Gamma}^K = 1.$

For the time discretization, as for the FDTD method, we use a Leap-Frog numerical scheme where the electric fields are evaluated at the time $n\Delta t$ and the magnetic fields at the time $(n + \frac{1}{2})\Delta t$, with Δt the time step and n the current iteration.

In order to define a set \mathcal{B} of basis functions of \mathbf{V}_r , we first define a set $\hat{\mathcal{B}}$ of basis functions of \hat{K} . Let $\hat{\mathbf{x}}_{ijk} = (\hat{x}_i, \hat{y}_j, \hat{z}_k)$, $1 \leq i, j, k \leq r+1$ be a set of points of \hat{K} , where \hat{x}_i, \hat{y}_j and \hat{z}_k are Gauss quadrature points on $[0, 1]$. At the point $\hat{\mathbf{x}}_{ijk}$, we define on \hat{K} three basis functions $\hat{\varphi}_{ijk}^l(\hat{x}, \hat{y}, \hat{z}) = L_i(\hat{x})L_j(\hat{y})L_k(\hat{z})e^l$ where $L_i(\hat{x}) = \frac{\prod_{m=1, m \neq i}^{r+1} (\hat{x} - \hat{x}_m)}{\prod_{m=1, m \neq i}^{r+1} (\hat{x}_i - \hat{x}_m)}$ is the Lagrange interpolation polynomial and

$(e^l)_{l=1,2,3}$ denotes the classical cartesian base. On $K \in \mathcal{T}$, the corresponding basis functions are defined by $\varphi_{ijk}^l \circ F_K(\hat{\mathbf{x}}) = (DF_K^*)^{-1} \hat{\varphi}_{ijk}^l(\hat{\mathbf{x}})$ where $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z})$. We say the DG method is a Q_r approximation when we choose a spatial approximation order of r .

Thanks to the chosen approximation space, we have:

$$\begin{cases} \int_K (\nabla \times \varphi_{ijk}^l) \cdot \varphi_{mnp}^s dx = \text{sign}(J_K) \int_{\hat{K}} (\hat{\nabla} \times \hat{\varphi}_{ijk}^l) \cdot \hat{\varphi}_{mnp}^s d\hat{x} \\ \int_{\partial K} (\varphi_{ijk}^l \times n_K) \cdot \varphi_{mnp}^s ds = \text{sign}(J_K) \int_{\partial \hat{K}} (\hat{\varphi}_{ijk}^l \times \hat{n}) \cdot \hat{\varphi}_{mnp}^s d\hat{s} \end{cases} \quad (4)$$

where \hat{n} is the outward unit normal to \hat{K} .

By using equations 4, a Gauss quadrature rule to evaluate integrals and the fact that the electric and magnetic fields are both expressed as

$$U(\mathbf{x}, t) = \sum_{l=1}^3 \sum_{ijk=1}^{r+1} U(t)_{ijk}^l \varphi_{ijk}^l(\mathbf{x}),$$

we obtain:

- For Mass matrices, $((M)_{pl})$ denotes the (p, l) component of the matrix M):

$$\int_K \varepsilon U \cdot \varphi_{ijk}^l dx = \omega_{ijk} \sum_{p=1}^3 U_{K,ijk}^p |J_K| (DF_K^{-1}(\varepsilon \circ F_K) DF_K^{*-1})_{pl}(\hat{x}_{ijk})$$

- For Stiffness matrices, $((M)_l)$ denotes the component l of vector M):

$$\int_K \nabla \times U \cdot \varphi_{ijk}^l = \text{sign}(J_K) \sum_{mnq=1}^{r+1} \sum_{p=1}^3 \omega_{ijk} U_{K,ijk}^p (\hat{\nabla} \times \hat{\varphi}_{ijk}^m(\hat{x}_{ijk}))_l$$

- For Jump matrices:

$$\begin{aligned} \int_{\partial K} \llbracket H \times n_K \rrbracket_{\partial K} \cdot \varphi_{ijk}^l ds &= -\text{sign}(J_{K'}) \int_{\partial \hat{K}} (DF_K^{-1} \circ S_{K'K} DF_{K'}) (\hat{H}_{K'} \times \hat{n}) \cdot \hat{\varphi}_{ijk}^l \circ S_{K'K} d\hat{s} \\ &+ \text{sign}(J_K) \int_{\partial \hat{K}} (\hat{H}_K \times \hat{n}) \cdot \hat{\varphi}_{ijk}^l d\hat{s}. \end{aligned}$$

In the above expressions ω_{ijk} is the quadrature weight at point \hat{x}_{ijk} and $S_{K'K} = F_K^{-1} \circ F_{K'}$. $DF_K^{-1} \circ S_{K'K} DF_{K'}$ is a permutation matrix constant per face (see [8] for more detail).

The DG formulation (3) finally leads to:

$$\begin{cases} M_\varepsilon \partial_t E + M_\sigma E = RH - S^i H, \\ M_\mu \partial_t H = -RE + S^i E + S^b E, \end{cases} \quad (5)$$

where M_ε , M_μ and M_σ are 3×3 block-diagonal matrices, R the stiffness matrix, S^i and S^b are jump matrices. Thanks to the choice of approximation space and basis functions, only the mass matrix has to be stored because of its dependance on the cells K . Stiffness and jump matrices just require to store the sign of the Jacobians J_K and some computations made on the reference element \widehat{K} .

The important advantage of this DG method is to give, regardless of the space approximation order, a very low memory storage and a small cost of computation to evaluate the matrices of the numerical scheme. This allows us to use meshes with a small number of cells and a high order spatial approximation to obtain very accurate solutions. Consequently, the memory and CPU costs are lower than for methods based on an order 2 spatial approximation using more refined meshes. In fact, high order spatial approximation and unstructured meshes reduce the dispersive and dissipative errors of the scheme, and improve the accuracy near the structures compared to staircase methods such as FDTD. Therefore the proposed method is well-adapted to Electromagnetic Compatibility (EMC) problems for which it is essential to know the fields near the structures, but also to cavity problems where the dispersive and dissipative errors cannot be neglected.

To illustrate these advantages, we consider the propagation of a mode inside a perfectly metallic cavity with an edge length equal to 1m. We study the propagative mode (3,0,0), whose analytical solution is given by:

$$\begin{cases} E_x = E_y = H_z = 0 \\ E_z = \sin(3\pi(x - x_0)) \sin(3\pi(y - y_0)) \cos(\omega t) \\ H_x = \frac{3\pi}{\omega\mu_0} \sin(3\pi(x - x_0)) \cos(3\pi(y - y_0)) \sin(\omega t) \\ H_y = \frac{3\pi}{\omega\mu_0} \cos(3\pi(x - x_0)) \sin(3\pi(y - y_0)) \sin(\omega t), \end{cases}$$

with $\omega = c_0 3\pi\sqrt{2}$ and (x_0, y_0, z_0) the center of the cavity. In figure 1 and table 1, we can compare results obtained with the FDTD and DG methods. The improvement is expressed in terms of storage and CPU time. In this example, with a high spatial approximation (Q_6) with the DG method, only $3 \times 3 \times 3$ cartesian cells have been required to mesh the cavity.

<i>Method</i>	<i>FDTD $\lambda/10$</i>	<i>FDTD $\lambda/20$</i>	<i>FDTD $\lambda/30$</i>	<i>FDTD $\lambda/40$</i>	<i>DG Q_5</i>	<i>DG Q_6</i>
<i>CPU time (s)</i>	40	120	300	840	257	490
<i>Memory (MBytes)</i>	1.5	5	12	27	2	2.7

Table 1

Memory and CPU time costs

Other examples have been treated in [8] and show the advantages of this DG

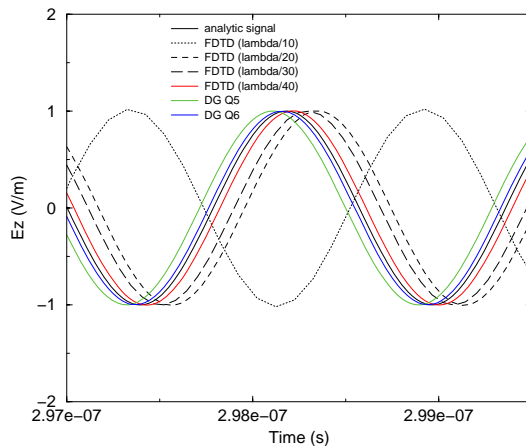


Fig. 1. Comparison of FDTD/DG results at the center of the cavity method for EM scattering problems.

3 Penalization of the centered fluxes by dissipative terms

For complex problems as usually encountered in industry, the meshes are built by using specific tools which, generally, do not generate automatic regular meshes. In particular, important differences between the sizes of the biggest and the smallest cells can be observed, as well as very distorted cells. In such cases, two problems appear:

- The use of a centered fluxes formulation can sometimes generate numerical spurious modes which degrade the accuracy of the solution.
- If strong cell-size disparities are encountered, it becomes necessary to use a small global time step which results in an important loss of efficiency of the scheme. In such cases, the advantages (accuracy and memory) of the DG method have a prohibitive cost: the computational time.

The second point will be treated in the next section by introducing local time-stepping strategies. For the first point, several studies [20][21] have been done to analyze the importance of the dispersive errors and spurious modes in DG schemes. The solution proposed here to reduce or eliminate spurious mode consists in adding some dissipative terms in the numerical scheme. Indeed, in the formalism (3) equivalent to the Maxwell equations, jump terms have been added on $E \times n$ and $H \times n$, and dissipative jump terms on $n \times (E \times n)$ and $n \times (H \times n)$. In the previous scheme we neglected the dissipative terms. A numerical study highlights their importance in the method when the meshes are very distorted [9]. In particular their ability to reduce considerably the spurious modes have to be mentioned. Then, we introduce these dissipative terms through penalization terms with two positive coefficients λ_{PE} , λ_{PH} without

modifying the time approximation. The new formulation of the problem on each element K becomes:

$$\begin{cases} \int_K \varepsilon \partial_t E_h \cdot \psi \, dx + \int_K \sigma E_h \cdot \psi \, dx = \int_K \nabla \times H_h \cdot \psi \, dx \\ + \int_{\partial K} \left(\beta_{\partial K}^K (\llbracket H_h \times n_K \rrbracket_{\partial K}^K + \text{sign}(\beta_{\partial K}^K) \lambda_{PE} \llbracket n_K \times (E_h \times n_K) \rrbracket_{\partial K}^K) \right) \cdot \psi \, ds, \\ \int_K \mu \partial_t H_h \cdot \phi \, dx = - \int_K \nabla \times E_h \cdot \phi \, dx \\ + \int_{\partial K} \left(\gamma_{\partial K}^K (\llbracket E_h \times n_K \rrbracket_{\partial K}^K + \text{sign}(\gamma_{\partial K}^K) \lambda_{PH} \llbracket n_K \times (H_h \times n_K) \rrbracket_{\partial K}^K) \right) \cdot \phi \, ds, \end{cases}$$

In this expression, the dissipative terms of the forms $\int_{\partial K} [n_K \times (v \times n_K)] \cdot \phi \, ds$ can be evaluated as follows:

$$\int_{\partial K} \llbracket n_K \times (v \times n_K) \rrbracket \cdot \phi \, ds = \int_{\partial K} n_K \times (v_{K'} \times n_K) \cdot \phi \, ds - \int_{\partial K} n_K \times (v_K \times n_K) \cdot \phi \, ds, \quad (6)$$

where v_K and $v_{K'}$ are the fields taken respectively to the cells K and K' , adjacent at the face ∂K . We obtain for the term with v_K :

$$\begin{aligned} \int_{\partial K} n_K \times (v_K \times n_K) \cdot \phi \, ds &= \int_{\partial K} (v_K \times n_K) \cdot (\phi \times n_K) \, ds \\ &= \int_{\partial \hat{K}} \frac{DF_K}{J_K \|DF_K^{*-1} \hat{n}_K\|} (\hat{v}_K \times \hat{n}_K) \cdot \frac{DF_K}{J_K \|DF_K^{*-1} \hat{n}_K\|} (\hat{\phi} \times \hat{n}_K) |J_K| \|DF_K^{*-1} \hat{n}_K\| \, d\hat{s} \\ &= \int_{\partial \hat{K}} \frac{1}{|J_K| \|DF_K^{*-1} \hat{n}_K\|} DF_K^* DF_K (\hat{v} \times \hat{n}_K) \cdot (\hat{\phi} \times \hat{n}_K) \, d\hat{s}, \end{aligned}$$

and for the term with $v_{K'}$:

$$\begin{aligned} \int_{\partial K} n_K \times (v_{K'} \times n_K) \cdot \phi \, ds &= \int_{\partial K} (v_{K'} \times n_K) \cdot (\phi \times n_K) \, ds \\ &= \int_{\partial \hat{K}} (v_{K'} \circ F_K(\hat{x}) \times n_K) \cdot \frac{DF_K}{J_K \|DF_K^{*-1} \hat{n}_K\|} (\hat{\phi} \times \hat{n}_K) |J_K| \|DF_K^{*-1} \hat{n}_K\| \, d\hat{s}. \end{aligned}$$

The maps $F_{K'}$ and F_K give $F_{K'}(\partial \hat{K}') = \partial K = F_K(\partial \hat{K})$. Consider the change of variable from $\partial \hat{K}$ to $\partial \hat{K}'$, by using $S_{K'K} = F_K^{-1} \circ F_{K'}$ on the previous integral, we obtain:

$$\begin{aligned} - \int_{\partial \hat{K}'} \frac{DF_{K'}}{J_{K'} \|DF_{K'}^{*-1} \hat{n}_{K'}\|} (\hat{v}_{K'} \times \hat{n}_{K'}) \cdot \frac{DF_K \circ S_{K'K}}{(J_K \|DF_K^{*-1} \hat{n}_K\|) \circ S_{K'K}} (\hat{\phi} \times \hat{n}_K) \circ \\ S_{K'K} (|J_K| \|DF_K^{*-1} \hat{n}_K\|) \circ S_{K'K} \, d\hat{s}', \end{aligned}$$

where $n_{K'} = -n_K$. By using:

$$J_{K'} \|DF_{K'}^{*-1} \hat{n}_{K'}\| = (J_K \|DF_K^{*-1} \hat{n}_K\|) \circ S_{K'K},$$

we finally obtain:

$$- \int_{\partial K'} \frac{DF_{K'}^*(DF_K \circ S_{K'K})}{\|J_{K'}\| \|DF_{K'}^{*-1} \times \hat{n}_{K'}\|} (\hat{v}_{K'} \times \hat{n}_{K'}) \cdot (\hat{\phi} \times \hat{n}_K) \circ S_{K'K} d\hat{s}'.$$

On the new expression of the jump integrals, we need to evaluate and store a 3×3 matrix $DF_K^* DF_K$ at each unknown of each surface of the cell K . Although this modification induces an increase of memory storage, the method remains advantageous because the others integrals in the formulation are not modified and need low memory storage.

3.1 L^2 -Stability of the spatial dissipative scheme

In [8] a stability result has been given for our spatial non-dissipative DG method. When dissipative terms are added, it is important to obtain a stable numerical method. In this subsection, we propose a condition to ensure the stability of our spatial dissipative DG method.

Recall the dissipative DG method :

$$\left\{ \begin{array}{l} \int_K \varepsilon_K \frac{E_h^{n+1} - E_h^n}{\Delta t} \cdot \psi \, dx = \int_K \nabla \times H_h^{n+\frac{1}{2}} \cdot \psi \, dx \\ + \int_{\partial K} \left(\beta [H_h^{n+\frac{1}{2}} \times n_K]_{\partial K}^K + \frac{\lambda}{Z_{\partial K}} [n_K \times (E_h^n \times n_K)]_{\partial K}^K \right) \cdot \psi \, ds, \\ \int_K \mu_K \frac{H_h^{n+\frac{1}{2}} - H_h^{n-\frac{1}{2}}}{\Delta t} \cdot \phi \, dx = - \int_K \nabla \times E_h^n \cdot \phi \, dx \\ + \int_{\partial K} \left(\gamma [E_h^n \times n_K]_{\partial K}^K + \frac{\lambda}{Y_{\partial K}} [n_K \times (H_h^{n-\frac{1}{2}} \times n_K)]_{\partial K}^K \right) \cdot \phi \, ds, \end{array} \right. \quad (7)$$

where $Z_\Gamma = \frac{1}{2} \left(\sqrt{\frac{\mu_K}{\varepsilon_K}} + \sqrt{\frac{\mu_{K'}}{\varepsilon_{K'}}} \right)$ and $Y_\Gamma = \frac{1}{2} \left(\sqrt{\frac{\varepsilon_K}{\mu_K}} + \sqrt{\frac{\varepsilon_{K'}}{\mu_{K'}}} \right)$ with $\Gamma = K \cap K'$ and considering $\lambda_{PE} = \frac{\lambda}{Z_{\partial K}}$, $\lambda_{PH} = \frac{\lambda}{Y_{\partial K}}$. To generalize the L^2 stability property, we take also a permittivity ε_k and a permeability μ_K constant by cell.

Taking up the L^2 -stability analysis carried out in [8], we easily obtain:

Proposition 1

$$\begin{aligned} \mathcal{E}_h^{n+1} - \mathcal{E}_h^n &= -\lambda\Delta t \sum_{\Gamma \in \mathcal{F}^i} \int_{\Gamma} \left(\frac{1}{Z_{\Gamma}} \llbracket E_h^n \times n_{\Gamma} \rrbracket \cdot \llbracket E_h^n \times n_{\Gamma} \rrbracket + \frac{1}{Z_{\Gamma}} \llbracket E_h^n \times n_{\Gamma} \rrbracket \cdot \llbracket E_h^{n+1} \times n_{\Gamma} \rrbracket \right. \\ &\quad \left. + \frac{1}{Y_{\Gamma}} \llbracket H_h^{n+\frac{1}{2}} \times n_{\Gamma} \rrbracket \cdot \llbracket H_h^{n+\frac{1}{2}} \times n_{\Gamma} \rrbracket + \frac{1}{Y_{\Gamma}} \llbracket H_h^{n-\frac{1}{2}} \times n_{\Gamma} \rrbracket \cdot \llbracket H_h^{n+\frac{1}{2}} \times n_{\Gamma} \rrbracket \right) \end{aligned}$$

where $\mathcal{E}_h^n = \sum_{K \in \mathcal{T}_h} \left(\int_K \varepsilon_K E_{h,K}^n \cdot E_{h,K}^n dx + \int_K \mu_K H_{h,K}^{n+\frac{1}{2}} \cdot H_{h,K}^{n-\frac{1}{2}} dx \right)$ and \mathcal{F}^i is the set of internal faces of the mesh \mathcal{T} ie if $\Gamma \in \mathcal{F}^i$ then $\exists K, K' \in \mathcal{T}_h$ such that $\Gamma = K \cap K'$.

Proof 1 *This proof is classical and does not raise any difficulty. That is why, we only give its sketch.*

In order to derive this result, we take the test functions

$$\psi = E_h^{n+1} + E_h^n \text{ and } \phi = H_h^{n+\frac{1}{2}}$$

Next, we test the first equation of (7) at time n and the second at times n and $n+1$. Finally, by adding these three equations for all cells of the mesh \mathcal{T} , we obtain the result.

In the case where the scheme is not dissipative (ie $\lambda = 0$) we obviously find the well-known discrete energy conservation ie $\mathcal{E}_h^{n+1} = \mathcal{E}_h^n$.

The estimate $ab \leq (a^2 + b^2)/2$ leads to:

$$\begin{aligned} \mathcal{E}_h^{n+1} - \mathcal{E}_h^n &\leq \frac{\lambda\Delta t}{2} \sum_{\Gamma \in \mathcal{F}^i} \left(-\frac{1}{Z_{\Gamma}} \|\llbracket E_h^n \times n_{\Gamma} \rrbracket\|_{0,\Gamma}^2 + \frac{1}{Z_{\Gamma}} \|\llbracket E_h^{n+1} \times n_{\Gamma} \rrbracket\|_{0,\Gamma}^2 \right. \\ &\quad \left. - \frac{1}{Y_{\Gamma}} \|\llbracket H_h^{n+\frac{1}{2}} \times n_{\Gamma} \rrbracket\|_{0,\Gamma}^2 + \frac{1}{Y_{\Gamma}} \|\llbracket H_h^{n-\frac{1}{2}} \times n_{\Gamma} \rrbracket\|_{0,\Gamma}^2 \right) \end{aligned} \tag{8}$$

where $\|\cdot\|_{0,X}$ is the L^2 norm on X .

So, one can define a better adapted discrete energy:

$$\tilde{\mathcal{E}}_h^n = \mathcal{E}_h^n - \frac{\lambda\Delta t}{2} \left(\|\llbracket E_h^n \times n_{\Gamma} \rrbracket\|_{0,\mathcal{F}^i,Z}^2 - \|\llbracket H_h^{n-\frac{1}{2}} \times n_{\Gamma} \rrbracket\|_{0,\mathcal{F}^i,Y}^2 \right)$$

with $\|\cdot\|_{0,\mathcal{F}^i,X}^2 = \sum_{\Gamma \in \mathcal{F}^i} \frac{1}{X_\Gamma} \|\cdot\|_{0,\Gamma}^2$. The estimate (8) becomes:

$$\tilde{\mathcal{E}}_h^{n+1} - \tilde{\mathcal{E}}_h^n \leq 0$$

To prove the L²-stability of the scheme, one must find a CFL condition for which $\tilde{\mathcal{E}}_h^n$ is a positive definite quadratic form [8].

Noting that

$$\tilde{\mathcal{E}}_h^n \geq \mathcal{E}_h^n - \frac{\lambda \Delta t}{2} \|[[E_h^n \times n_\Gamma]]\|_{0,\mathcal{F}^i,Z}^2$$

and that

$$\begin{aligned} \|[[E_h^n \times n_\Gamma]]\|_{0,\mathcal{F}^i,Z}^2 &= \sum_{\Gamma=K \cap K' \in \mathcal{F}^i} \frac{2}{\sqrt{\frac{\mu_K}{\varepsilon_K}} + \sqrt{\frac{\mu_{K'}}{\varepsilon_{K'}}}} \|[[E_h^n \times n_\Gamma]]\|_{0,\Gamma}^2 \\ &\leq \sum_{\Gamma=K \cap K' \in \mathcal{F}^i} \frac{4}{\sqrt{\frac{\mu_K}{\varepsilon_K}} + \sqrt{\frac{\mu_{K'}}{\varepsilon_{K'}}}} \left(\|E_{h,K}^n \times n_\Gamma\|_{0,\Gamma}^2 + \|E_{h,K'}^n \times n_\Gamma\|_{0,\Gamma}^2 \right) \\ &\leq 4 \sum_{\Gamma=K \cap K' \in \mathcal{F}^i} \left(\frac{c_K}{1 + \sqrt{\frac{\varepsilon_K \mu_{K'}}{\varepsilon_{K'} \mu_K}}} \|E_{h,K}^n \times n_\Gamma\|_{0,\Gamma}^2 + \frac{c_{K'}}{1 + \sqrt{\frac{\varepsilon_{K'} \mu_K}{\varepsilon_K \mu_{K'}}}} \|E_{h,K'}^n \times n_\Gamma\|_{0,\Gamma}^2 \right) \\ &\leq 4 \sum_{K \in \mathcal{T}_h} c_K \|\tilde{E}_{h,K}^n \times n_K\|_{0,\partial K^i}^2 \quad (\partial K^i = \text{the set of faces of } K \text{ belonging to } \mathcal{F}^i) \\ &\leq 4 \lambda_{max} \left(\hat{\mathcal{D}}^{-\frac{1}{2}} \hat{\mathcal{B}} \hat{\mathcal{D}}^{-\frac{1}{2}} \right) \sum_{K \in \mathcal{T}_h} c_K \|\hat{E}_K\|_{0,\hat{K}}^2 \text{ see [8]} \end{aligned}$$

where $\lambda_{max}(A)$ corresponds to the greatest eigenvalue of the matrix A , $c_K = 1/\sqrt{\varepsilon_K \mu_K}$, $\tilde{E}_{h,K}^n = E_{h,K}^n \sqrt{\varepsilon_K}$ and $\hat{\mathcal{D}}$, $\hat{\mathcal{B}}$ are the $3(r+1)^3 \times 3(r+1)^3$ matrices defined by: $\forall l, l' \in \{1, 2, 3\}$ and $\forall I = (i, j, k)$, $I' = (i', j', k') \in \{1, \dots, r+1\}^3$

$$\hat{\mathcal{D}}((l, I), (l', I')) = \delta_{ll'} \delta_{II'} \omega_{ijk}$$

$$\hat{\mathcal{B}}((l, I), (l', I')) = \int_{\partial \hat{K}} (\hat{\varphi}_{ijk}^l \times \hat{n}) \cdot (\hat{\varphi}_{i'j'k'}^{l'} \times \hat{n}) d\hat{s},$$

it is easy to see that the stability condition in [8] becomes:

$$\frac{\Delta t}{\Lambda_K} < \frac{2}{c_K} \frac{1}{\sqrt{\lambda_{\max}(\hat{\mathcal{D}}^{-\frac{1}{2}} \hat{\mathcal{R}} \hat{\mathcal{D}}^{-\frac{1}{2}})} + \left(\frac{1}{2} \max_{1 \leq i \leq \text{nbfi}_K} \left(\sqrt{\frac{\mu_K}{\mu_{V(i,K)}}}, \sqrt{\frac{\varepsilon_K}{\varepsilon_{V(i,K)}}} \right) + 4\lambda\right) \lambda_{\max}(\hat{\mathcal{D}}^{-\frac{1}{2}} \hat{\mathcal{B}} \hat{\mathcal{D}}^{-\frac{1}{2}})}$$

where $\Lambda_K = \min_{1 \leq i, j, k \leq r+1} \left(\frac{|J_K(\hat{\mathbf{x}}_{ijk})|}{\lambda_{\max}((DF_K^* DF_K)(\hat{\mathbf{x}}_{ijk}))} \right)$ (in the case of a uniform cartesian grid whose the spatial step is h , $\Lambda_K = h$).

So, the dissipative DG method using a leap-frog approximation in time is L^2 -stable. As we have noted in practice, we can see that this condition is slightly more restrictive than the one obtained with the non-dissipative scheme. This is due to the backward discretization for the time approximation of penalization terms.

3.2 *A-priori error estimate*

The advantage of the proposed spatial dissipative scheme is significant with strongly distorted meshes. Particularly, by a mathematical analysis we can demonstrate a gain in the order of convergence of the scheme which implies the convergence for all spatial orders of approximation.

In [10], a complete error analysis of the non-dissipative scheme has been carried out. In particular, we have pointed out that the choice of the approximation (Hexahedrals, centered flux and the presence of the Jacobian matrix in \mathbf{V}_r) can imply a loss of spatial convergence and the Q_1 scheme can even become no-convergent. By adding the penalization terms, a similar study can be made on the convergence of the dissipative scheme.

Let (E, H) and (E_h, H_h) be respectively the exact solution of the Maxwell equations and the approximate DG solution belonging to V_r . Then, to evaluate a convergence of the DG method, we evaluate an overestimation of $\|(E - E_h, H - H_h)\|_*$ by a term depending of the spatial step size and the order of the DG scheme. The $\|(\cdot, \cdot)\|_*$ is the energy norm defined by $\|(E, H)\|_*^2 = \int_{\Omega} (\varepsilon E \cdot E + \mu H \cdot H) dx = \|E\|_{0,\varepsilon,\Omega}^2 + \|H\|_{0,\mu,\Omega}^2$, where $\|E\|_{0,\varepsilon,\Omega}^2 = \int_{\Omega} (\varepsilon E \cdot E) dx$ and $\|H\|_{0,\mu,\Omega}^2 = \int_{\Omega} (\mu H \cdot H) dx$.

Consider $(u, w) \in V_r \times V_r$, we write $E - E_h = E - v + v - E_h = \Delta_E^P - \Delta_E^I$ and $H - H_h = H - w + w - H_h = \Delta_H^P - \Delta_H^I$ with $\Delta_E^P = E - v$, $\Delta_E^I = E_h - v$, $\Delta_H^P = H - w$ and $\Delta_H^I = H_h - w$.

We have the two following propositions:

Proposition 2 Let $(v, w) \in \mathbf{V}_r \times \mathbf{V}_r$ be the solution of the problem: $\forall (v', w') \in \mathbf{V}_r \times \mathbf{V}_r$ and $\forall K \in \mathcal{T}$,

$$\begin{aligned} \int_K \varepsilon v \cdot v' dx + \int_K \nabla \times w \cdot v' dx - \int_{\partial K} \left(\beta \llbracket w \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (v \times n) \rrbracket_{\partial K}^K \right) \cdot v' ds &= l_1(v') \\ \int_K \mu w \cdot w' dx - \int_K \nabla \times v \cdot w' dx - \int_{\partial K} \left(\gamma \llbracket v \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (w \times n) \rrbracket_{\partial K}^K \right) \cdot w' ds &= l_2(w') \end{aligned} \quad (9)$$

where l_1, l_2 are the two linear forms on \mathbf{V}_r defined by:

$$\begin{aligned} l_1(v') &= \int_K \varepsilon E \cdot v' dx + \int_K \nabla \times H \cdot v' dx \\ l_2(w') &= \int_K \mu H \cdot w' dx - \int_K \nabla \times E \cdot w' dx. \end{aligned}$$

Then, we have:

$$\frac{d}{dt} \|(\Delta_E^I, \Delta_H^I)\|_* \leq \|\Delta_{\frac{\partial E}{\partial t}}^P\|_{0,\varepsilon,\Omega} + \|\Delta_{\frac{\partial H}{\partial t}}^P\|_{0,\mu,\Omega} + \|\Delta_E^P\|_{0,\varepsilon,\Omega} + \|\Delta_H^P\|_{0,\mu,\Omega} \quad (10)$$

Proposition 3 If we assume that the exact solution verifies $(E, H) \in H^{s+1}(\mathcal{T})$ for $s \geq 0$, then there exists a constant $C > 0$ such that

$$\|(\Delta_E^P, \Delta_H^P)\|_* \leq Ch^{\min(s-\frac{1}{2}, r-\frac{1}{2})} \max(\|E\|_{s+1,h}, \|H\|_{s+1,h})$$

where $H^s(\mathcal{T}) = \{v \in [L^2(\Omega)]^3 : \forall K \in \mathcal{T}, v|_K \in [H^s(K)]^3\}$ and $\|v\|_{s,h}^2 = \sum_{K \in \mathcal{T}} \|v\|_{s,K}^2$.

The technical proofs of these two propositions are given at the appendix A. By using the proposition (2) and (3) and the Gronwall lemma on the time interval $(0, T)$, we have:

Theorem 1 Let r be a positive integer. Assume that the exact solution verifies $(E, H) \in H^{s+1}(\mathcal{T})$ and $(\frac{\partial E}{\partial t}, \frac{\partial H}{\partial t}) \in H^{s'+1}(\mathcal{T})$ for $s, s' \geq 0$ real and $0 < h_K \leq 1$. Then, we have the global estimate of the interpolation error:

$$\|(\Delta_E^I, \Delta_H^I)\|_*(T) \leq \|(\Delta_E^I, \Delta_H^I)\|_*(0) + CT h^{\min(s-\frac{1}{2}, s'-\frac{1}{2}, r-\frac{1}{2})} A(T, E, H) \quad (11)$$

where

$$A(T, E, H) = \max_{t \in (0, T)} \left(\|E\|_{s+1,h}(t), \|H\|_{s+1,h}(t), \left\| \frac{\partial E}{\partial t} \right\|_{s'+1,h}(t), \left\| \frac{\partial H}{\partial t} \right\|_{s'+1,h}(t) \right).$$

Finally, by using (11) and (3), we deduce the error of the DG-scheme by:

$$\|(E - E_h, H - H_h)\|_*^2 \leq 2(\|(\Delta_E^P, \Delta_H^P)\|_*^2 + \|(\Delta_E^I, \Delta_H^I)\|_*^2).$$

In conclusion, if the exact solution is smooth enough, the convergence rate for the penalized scheme is $r - \frac{1}{2}$ versus $r - 1$ for the non-dissipative scheme. So, the dissipative terms ensure the L^2 -convergence for the Q_1 approximation.

3.3 Some numerical results

In [10], we have proved that the value $r - 1$ for the spatial convergence rate (in L^2 norm) of non-dissipative scheme seems to be optimal for the general unstructured meshes. The previous error analysis shows that this bound becomes $r - 1/2$ when one adds dissipative terms. These two results are sufficient to affirm that the dissipative scheme is better convergent than the non-dissipative one. In this part, we just illustrate this result by some numerical results.

To underline the convergence results for the Q_1 approximation, we give two examples comparing the dissipative and the non-dissipative approaches. The first example is the previous cavity problem treated in section 2. For several non-regular unstructured meshes of the cavity with an assimilate decreasing spatial step size h , we compare in table 2 the errors in norm L^2 between the analytic solution and the solutions obtained with the dissipative and the non-dissipative Q_1 spatial approximation. We can see in this table a better

Method	without dissipative terms		with dissipative terms	
	L^2 error	order	L^2 error	order
$h = 1cm$	0.03699	×	0.02748	×
$h = 0.5cm$	0.01994	0.891	0.01369	1
$h = 0.25cm$	0.01037	0.943	0.00729	0.909
$h = 0.16cm$	0.00927	0.251	0.00497	0.858

Table 2

Error between numerical and analytical solutions for different spatial steps

convergence of the scheme by considering dissipative terms than without these terms. Figure 2 shows also an accuracy comparison between the solutions obtained with the two schemes for a given mesh of the cavity. We note in this figure, on the first curves, the advantage of the dissipative approach. This advantage is clearly due, as pointed out in the second curves of the figure, to the dispersive error induced by the non-dissipative scheme. Indeed, we can show by a mathematical study in one dimension that the dissipative scheme

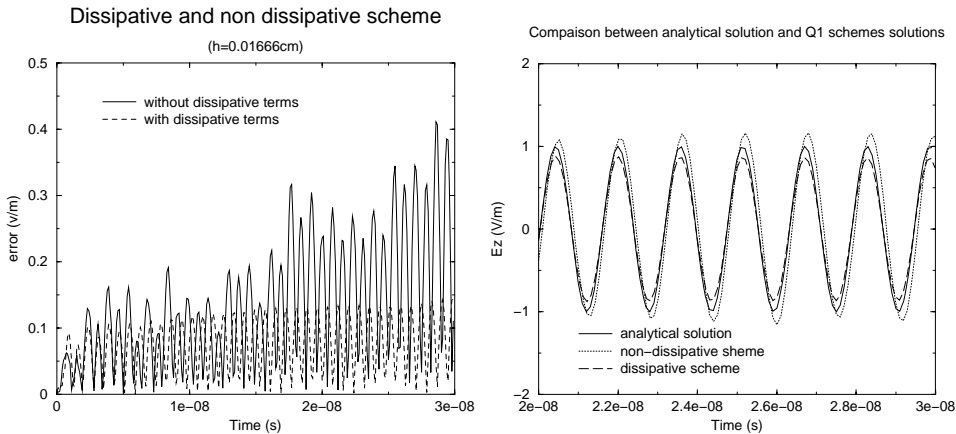


Fig. 2. Comparison between non-dissipative and dissipative schemes

is less dispersive than the non-dissipative scheme (order 4 instead of order 2 [13] [8]).

This cavity example shows the advantages to take into account the dissipative terms in our formalism when the mesh is non-regular, but for this kind of problem, a Q_1 approximation is not the best choice and an approximation of higher order is more appropriate in order to also avoid dissipative errors. However, for scattering problems, the Q_1 approximation can sometimes be sufficient and taking into account the dissipative terms improves the efficiency of the method. The second example proposed in this section gives an application related to this kind of problems. The problem consists in evaluating at a given test-point the electromagnetic fields scattered by a perfectly metallic sphere (center= $(0, 0, 0)$ and radius= 0.5m) illuminated with an incident plane wave given by $E_y(t, x, y, z) = 377 \exp\left(-\left(\frac{t + \frac{z-1}{3.68} - 1e-8}{5e-9}\right)^2\right)$. In figure 3, we compare the results obtained by using or not the dissipative terms in the scheme. We can see different solutions obtained with the non-dissipative Q_1 and Q_2 approximations, the dissipative Q_1 approximation and the FDTD method, being assumed here as the reference solution. The FDTD solution has been obtained for a mesh where the cell size is less than the smallest wavelength of the source spectrum divided by 80. This solution is quasi-similar for meshes with smaller cell sizes. We can notice that the solution obtained with the non-dissipative Q_1 approximation has a strange behavior. The use of a Q_2 approximation in the non-dissipative scheme improves the solution but remains less accurate than the solution obtained with the dissipative Q_1 approximation which is similar to the FDTD solution. Indeed, little oscillations appears in the Q_2 -approximation results due certainly to spurious modes. In fact, for scattering problems, the advantage of using dissipative terms in our DG scheme on non-regular meshes is double: as we obtain more accurate solutions with low spatial order approximation, less computational time is required to obtain equivalent solutions.

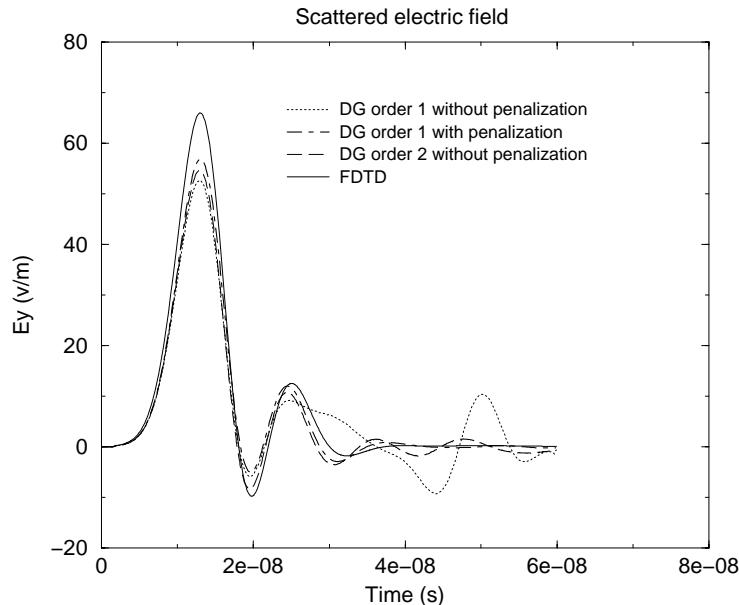


Fig. 3. Electric fields located at a given test-point $A = (1, 0, 0)$

4 Local time-stepping strategy based upon a leap-frog time scheme

In the numerical method proposed in this paper, the time domain has been approximated by using an explicit well known leap-frog scheme in the electromagnetism community [2][15], which permits to obtain a fast and sufficiently accurate (order 2) method of integration to treat a lot of problems. In the unstructured meshes used in our DG approach, we can have an important difference in terms of size between each cell. To ensure stability, the time step is given by the smallest cell. This can be an important constraint which implies an important increase of CPU-time for the simulations. To avoid this problem, a solution could be to take an implicit scheme to discretize the time. But, for Maxwell 3D-problems, the memory cost due to the storage of inverse mass-matrices are too important to consider using such an approach. Then, another solution is to use a local time-stepping strategy in our explicit scheme. Indeed, the time step imposed by the smallest cell is not necessary for all cells to ensure the stability of the scheme in the computational domain. In the literature, for Maxwell's equations, we can find local time-stepping methods which ensure or not a condition of stability. In particular, a method based on a FDTD scheme has been proposed in [14] to ensure an energy conservation. But, for 3D-Maxwell's equations, this method leads to a numerical scheme too expensive in terms of memory and computational time. Consequently, local time-stepping strategies based upon interpolations and fully explicit schemes, where a stability condition is difficult to be proved, remain the most attractive. In this paper, we focus our investigations on a method of this last family to improve our DG method.

Starting from our DG scheme with leap-frog time discretization, we proposed in [8] a 2-classes method using interpolations. In this approach, the computational domain is split into two parts: the first one composed of cells evaluated with a time step adapted to the smallest cells of the mesh, and the second one on which cells are evaluated with a larger time step. The largest time step is taken as a multiple of the smallest one in order to ensure coincidence at each step of the process, and interpolations are used to approximate unknown fields. This method gives good results and the computational time is considerably reduced with meshes composed of small and big cells. However, in general, the range of cells size have a continuous progression from smallest to biggest ones and, in the case of strongly refined unstructured meshes, it is widely required to use more than two classes of cells to reduce the CPU time cost of simulations. Unfortunately, using interpolations with more than two classes is too expensive in term of computational time and multi-class strategies must be investigated.

In this kind of multi-class methods, a non-dissipative DG approach, proposed by S. Piperno [15], was presented as symplectic [16], and therefore was supposed to conserve an energy quantity. It is based on the Verlet scheme, which is a reorganization of the classical leap-frog scheme into three steps. Let E^n and H^n be the electric and magnetic fields at the time step n , the values of the fields E^{n+1} and H^{n+1} at the time $n + 1$ by using the Verlet scheme are given by :

$$\begin{cases} H^{n+\frac{1}{2}} = H^n - \frac{\Delta t}{2} M_\mu^{-1} S E^n, \\ E^{n+1} = E^n + \Delta t M_\epsilon^{-1} S H^{n+\frac{1}{2}}, \\ H^{n+1} = H^{n+\frac{1}{2}} - \frac{\Delta t}{2} M_\mu^{-1} S E^{n+1}. \end{cases} \quad (12)$$

where Δt , M_ϵ , M_μ and S are respectively the time step, the mass matrices for electric and magnetic equation and the matrix related to the curl operator.

In his multi-class approach, Piperno bulks the cells into N sets or classes $1, 2, \dots, N-1, N$ which are associated to the respective time-steps $\frac{\Delta t}{2^{N-1}}, \frac{\Delta t}{2^{N-2}}, \dots, \frac{\Delta t}{2}, \Delta t$. Then, the smallest cells are in class 1 and the largest cells in class N . In the process, for an evaluation of the fields at a step Δt , the number of field evaluations inside each class is different. In figure 4 the steps are labeled in the order in which they are executed in the process.

In the figure 4, we can notice also that the multi-class approach can be defined at each time step Δt by a recursive process labeled $R^N(\Delta t)$ and given by :

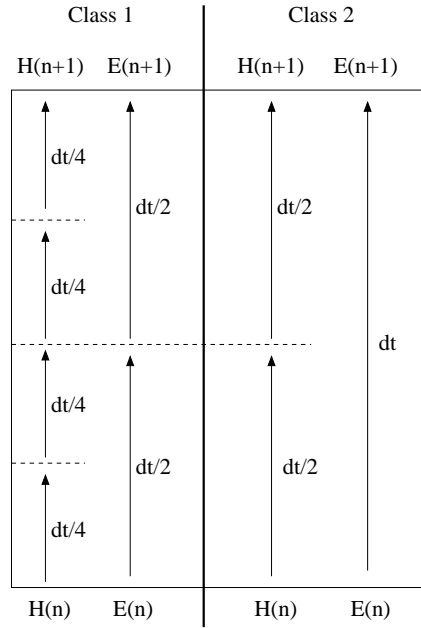


Fig. 4. Operations in a time step of the Piperno scheme with 2 classes

$$\left\{ \begin{array}{l} \text{Evaluate } R^{N-1} \left(\frac{\Delta t}{2} \right) \\ \text{Evaluate the cells into class } N \text{ by using equations (12)} \\ \text{Evaluate } R^{N-1} \left(\frac{\Delta t}{2} \right) \end{array} \right. \quad (13)$$

with the convention that R^0 performs no operation.

Note that this time domain scheme consists in a recursive call of the Verlet scheme on different classes (for $N = 1$, we retrieve the original Verlet scheme). However, even if the Verlet scheme is symplectic, we can not *a priori* affirm that this recursive scheme is symplectic too. Indeed, calling the Verlet scheme on different classes requires values of the fields at some unknown times. To palliate this difficulty, the last known fields available are used. Consequently, the scheme which is recursively called in (13) is not exactly the Verlet one, and the symplectic and stability properties of this multi-classes method are still an open question, even if an energy conservation is proved in [15] for $N = 2$. In particular, for long time simulations, it can be necessary to reduce the time steps in order to avoid instabilities. Nevertheless, this recursive scheme remains attractive. Indeed, regardless of the number of classes N , this scheme is easy to implement, fully explicit, it does not need any additional storage and gives good results: the numerical solution is comparable to standard ones and the computation time is significantly reduced. Thanks to the time-coincidence of fields computed with the Verlet scheme, the writing of a Verlet-based recursive scheme is natural.

From the same idea, we present here a leap-frog based recursive method,

better adapted to our scheme and more efficient in terms of computational time.

4.2 Recursive leap-frog method

To make comparison with the Piperno method, we describe the recursive leap-frog method in the case of a spatial non-dissipative formulation. To take into account a spatial dissipative formulation, we only need to add new jump terms which are split like the non-dissipative jump terms. This does not induce any particular difficulty and the analysis done in this section on comparison between the different methods remains the same. Indeed, for dissipative or non-dissipative formulation, the number of classes defined depends on the shape of the cell and of the variation of the time step in the set of cells. This last quantity is approximately the same for the two formulations.

For practical reasons, we rewrite our Discontinuous Galerkin formulations defined by the equations (5) under the equivalent form [15]:

$$\begin{cases} M_\varepsilon \partial_t E + M_\sigma E = A_H H - S^{i-} H = S H \\ M_\mu \partial_t H = -A_E E + S^{i-} E = -S^T E, \end{cases} \quad (14)$$

by splitting in (5), the jump terms S^i and $(S^i)^T$ in a cell into two parts :

- part inside the cell itself : S^{i+} and $(S^{i+})^T$
- part outside the cell S^{i-} and $(S^{i-})^T$

Then we obtain $A_E = R + S^{i+}$ and $A_H = R - (S^{i+})^T - S^b$. For the following, we consider $M_\sigma = 0$ to simplify the expressions but there is no difficulty to take into account this term on the recursive leap-frog method. As for the previous scheme, the cells of the mesh are grouped into N integration classes i , associated to the time step $(2m+1)^{N-i} \Delta t_{min}$, where m is a strictly positive integer and Δt_{min} the time step corresponding to the smallest cell of the mesh. In the sequel, we consider the case $m = 1$, which means that there is a factor 3 between the time steps of consecutive classes. We recall hereafter the expression of the leap-frog scheme with a time-step Δt at the step n is given by:

$$\begin{cases} M_\mu \frac{H^{n+1/2} - H^{n-1/2}}{\Delta t} = -S^T E^n \\ M_\varepsilon \frac{E^{n+1} - E^{n-1}}{\Delta t} = S H^{n+1/2}; \end{cases} \quad (15)$$

For $N = 2$, by considering (14), we propose the following multi-class leap-frog

method at cells located at the interface between class 1 and 2 can be written:

$$\left\{ \begin{array}{l} M_2^\mu \frac{H_2^{n+\frac{1}{2}} - H_2^{n-\frac{1}{2}}}{\Delta t} = -A_2 E_2^n + S_{21}^T E_1^n \\ M_1^\mu \frac{H_1^{n+\frac{1}{6}} - H_1^{n-\frac{1}{6}}}{\Delta t/3} = -A_1 E_1^n + S_{12}^T E_2^n \\ M_1^\varepsilon \frac{E_1^{n+\frac{2}{6}} - E_1^n}{\Delta t/3} = A_1 H_1^{n+\frac{1}{6}} - S_{12} H_2^{n+\frac{1}{2}*} \\ M_1^\mu \frac{H_1^{n+\frac{1}{2}} - H_1^{n+\frac{1}{6}}}{\Delta t/3} = -A_1 E_1^{n+\frac{2}{6}} + S_{12}^T E_2^{n*} \\ M_2^\varepsilon \frac{E_2^{n+1} - E_2^n}{\Delta t} = A_2 H_2^{n+\frac{1}{2}} - S_{21} H_1^{n+\frac{1}{2}} \\ M_1^\varepsilon \frac{E_1^{n+\frac{4}{6}} - E_1^{n+\frac{2}{6}}}{\Delta t/3} = A_1 H_1^{n+\frac{1}{2}} - S_{12} H_2^{n+\frac{1}{2}} \\ M_1^\mu \frac{H_1^{n+\frac{5}{6}} - H_1^{n-\frac{1}{2}}}{\Delta t/3} = -A_1 E_1^{n+\frac{4}{6}} + S_{12}^T E_2^{n+1*} \\ M_1^\varepsilon \frac{E_1^{n+1} - E_1^{n+\frac{4}{6}}}{\Delta t/3} = A_1 H_1^{n+\frac{5}{6}} - S_{12} H_2^{n+\frac{1}{2}*} \end{array} \right. \quad (16)$$

where in the matrices, the subscripts 1, 2, 21 and 12 are identified respectively to cells in class 1, 2 and terms in jumps coming from cells located in class 1 (respectively 2) into cells located in class 2 (respectively 1). For cells which are not located at the interface between cells 1 and 2, there is no problem and the classical leap-frog is applied (with the time-step of the class). Note that, as in (13), we replace fields at unknown times by the last known values, denoted * in (16). In figure 5 we give the operations proceed in a step of the multi-class Leap-frog method for $N = 3$.

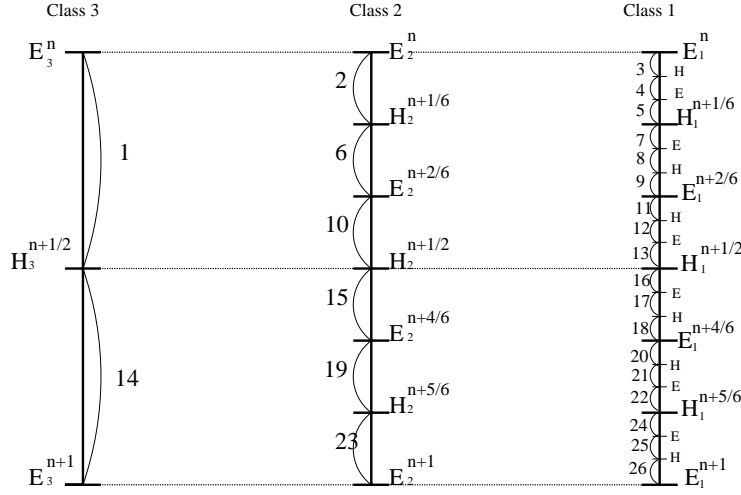


Fig. 5. Steps of recursive leap-frog scheme with 3 classes

More generally, if we label $LeapFrogH(n, \Delta t)$ (resp. $LeapFrogE(n, \Delta t)$) the first (resp. the second) equation of (15) applied to the cells belonging at the class N with a time-step Δt , we can define the multi-class leap-frog method

as a recursive process. A step of integration of the recursive leap-frog method is defined by:

$$\begin{cases} 1. \text{ Compute}H(N, \Delta t) \\ 2. \text{ Compute}E(N, \Delta t) \end{cases} \quad (17)$$

where the recursive functions $\text{Compute}H(N, \Delta t)$ and $\text{Compute}E(N, \Delta t)$ are respectively defined by:

$$\begin{array}{l} \underline{\text{Compute}H(N, \Delta t)} : \\ \left\{ \begin{array}{l} - \text{LeapFrog}H(N, \Delta t) \\ - \text{Compute}H\left(N - 1, \frac{\Delta t}{3}\right) \\ - \text{Compute}E\left(N - 1, \frac{\Delta t}{3}\right) \\ - \text{Compute}H\left(N - 1, \frac{\Delta t}{3}\right) \end{array} \right. \end{array} \quad \begin{array}{l} \underline{\text{Compute}E(N, \Delta t)} : \\ \left\{ \begin{array}{l} - \text{LeapFrog}E(N, \Delta t) \\ - \text{Compute}E\left(N - 1, \frac{\Delta t}{3}\right) \\ - \text{Compute}H\left(N - 1, \frac{\Delta t}{3}\right) \\ - \text{Compute}E\left(N - 1, \frac{\Delta t}{3}\right) \end{array} \right. \end{array}$$

with $\text{Compute}H(1, \delta t)$ defined by $\text{LeapFrog}H(1, \delta t)$ and $\text{Compute}E(1, \delta t)$ defined by $\text{LeapFrog}E(1, \delta t)$, where δt denotes the time-step.

Since the leap-frog is composed of only two steps (three for the Verlet scheme), this method requires 33% less computation than the Verlet-based recursive scheme, with the same advantages: the scheme is fully explicit, easy to write, does not require additional storage and gives good results with a CPU time significantly reduced. However, we have also the same problems for the stability study as in the Verlet-based method: the CFL must sometimes be strengthened for long-time simulations to ensure stability.

Various numerical validation results are given in section 4.3. This local time-stepping method, shows its interests on real problems and gives good comparison results with the two others local time-stepping strategies describe in this paper. The results obtained in this paper are given with a dissipative approach, but similar conclusion would be given for a non-dissipative approach.

4.3 Numerical Results

In this section, we present some numerical results obtained with the previous local time-stepping methods applied on three different meshes given in figure 6. These unstructured meshes, which are typical of problems encountered in industry, contain a large number of cells and present strong cell-size disparities. In these simulations, the aircraft, the space shuttle and the generic missile are all illuminated by a plane wave.

The proposed leap-frog multi-class local time-stepping method (R-LF) has

been compared with the 2-class method (2-CL) presented in [8] and the method (R-V) proposed by S. Piperno. We give in tables 3, 4 and 5 the repartition of cells by classes respectively obtained with the three methods. We can notice, for all the methods, the low percentage of small cells, but also important cell-size disparities. For example, there is at least a factor 2^{10} between small and largest cells of the missile mesh (because the R-V scheme uses 10 classes of cells). This explains the real efficiency of multi-class recursive methods on such meshes.

For the three meshes under consideration, table 6 shows the computational time gain obtained between the standard leap-frog scheme (without local time-stepping) and the three local time-stepping methods discussed in this paper. We first note that the R-LF method is always faster than the other methods. We can also remark that the gain obtained with the 2-CL method is limited whereas, with multi-class methods, the stronger the cell size disparities, the greater the gain in terms of computational time. For example, considering the missile mesh (which presents the most refined unstructured mesh), the improvement obtained with recursive methods is very significant: the R-LF method leads to a scheme 15 times faster than without local time-stepping. In such cases, the requirement of multi-class strategies is clearly highlighted.

Scheme \ Class	1	2	3	4	5	6	7
2-CL	1000	179000	×	×	×	×	×
R-LF	1000	16600	110400	42200	8900	×	×
R-V	300	2300	10800	56200	67700	34000	8700

Table 3

Cell repartition by classes for the aircraft mesh (~ 180000 cells)

Scheme \ Class	1	2	3	4	5	6
2-CL	200	105200	×	×	×	×
R-LF	100	6600	91000	7700	×	×
R-V	70	500	4000	38100	57300	5500

Table 4

Cell repartition by classes for the shuttle mesh (~ 105000 cells)

Sc \ Cl	1	2	3	4	5	6	7	8	9	10
2-CL	12	91000	×	×	×	×	×	×	×	×
R-LF	10	200	1400	14300	71600	3500	×	×	×	×
R-V	8	16	160	550	1500	5800	46000	33500	3300	200

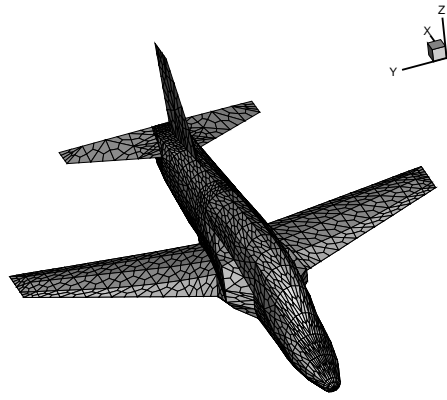
Table 5

Cell repartition by classes for the missile mesh (~ 91000 cells)

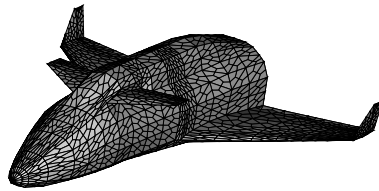
Scheme \ Mesh	Plane	Shuttle	Missile
2-CL	2.5	3.6	2.7
R-V	4.5	4.0	11.0
R-LF	5.5	6.0	15.0

Table 6

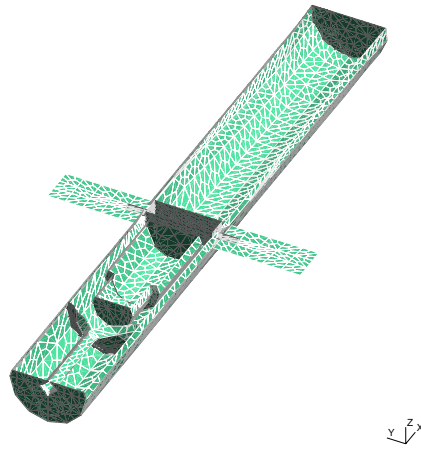
CPU-time gain obtained with different local time-stepping methods versus the leap-frog scheme (without local time-stepping)



(a) Aircraft



(b) Hermes space shuttle



(c) Generic missile (opened for display requirements)

Fig. 6. Meshes considered for comparisons of the different methods

On the previous example, the efficiency in terms of CPU-time reduction is clearly shown. In the next example, we compare the solutions obtained with and without the local time-stepping strategy. The example chosen is a perfectly conductive cube illuminated by a plane wave and the comparisons between the two approaches are made at different given test-points. Table 7 gives the results in terms of error between the solutions obtained for the two approaches and in terms of factor of gain by using the local time-stepping strategy rather the approach without local time-stepping. These results are given for different configurations by using meshes where the time step dt taken in the non local time-stepping strategy to ensure stability, decreases. We can see on this table that the convergence behavior in time of the two methods is the same and that the accuracy of the solution is conserved by using the local time-stepping strategy proposed.

Configuration	CPU-time factor	L ₂ error
dt=2.e-11s	1.5	1.63e-2
dt=1.e-11s	1.64	0.773e-2
dt=4.e-12s	1.51	0.69e-2

Table 7

Comparison in terms of CPU time and accuracy of the solution between the approaches using or not the local time-stepping strategy proposed

We have also compared the DG method using or not the local time-stepping strategy with the FDTD method. Figure 7 presents on the aircraft example an evaluated scattered component of the field taken at a point outside the aircraft by using the three methods We can see on this figure, the good agreement between the different solutions. In particular, the accuracy of the solution is not altered by using the local time-stepping strategy.

The advantages of using a local time-stepping method are clearly shown for scattering problems, but there remains some difficulties for cavity problems. Indeed, in such cases, long computation time are necessary and we cannot give a stability criteria for the proposed local time-stepping method yet. However, experimental results show that a suitable reduction of the smallest time step (CFL condition) increases the possible observation time and offers the possibility to use the method to treat these kinds of problems in a sufficient time of observation. For example, considering the cavity problem treated in section 2, if we strengthen the CFL by a factor 0.8, we obtain a stable solution for a duration time larger than the simulation time generally needed for electromagnetic industrial problems (see figure 8). Moreover, despite this reduction of the CFL condition, the use of local time-stepping in this kind of problems, remains more advantageous in terms of computational time than the FDTD method where the dispersive errors destroys the accuracy of the solution.

Finally, figure 9 presents a comparison between the DG method by using the

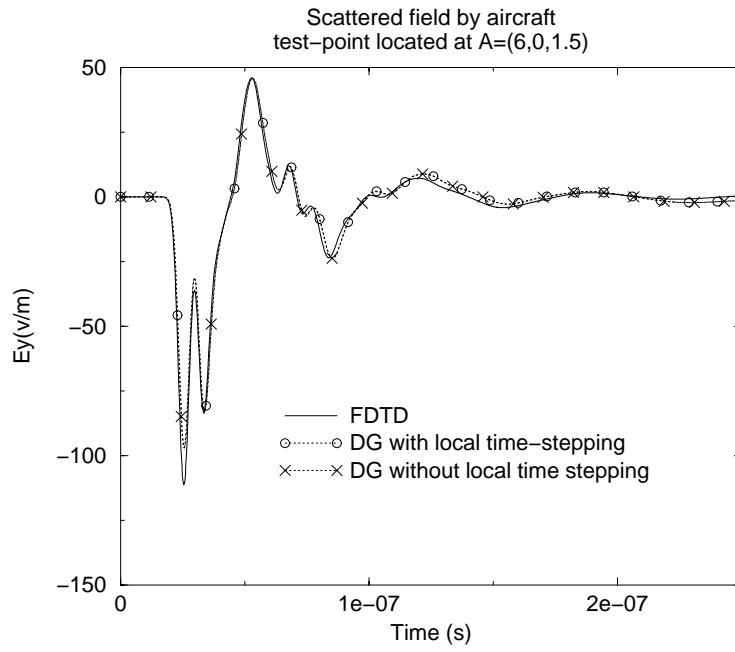


Fig. 7. Comparison of the accuracy of the solution by using or not the local time-stepping method with a Q_2 dissipative spatial approach

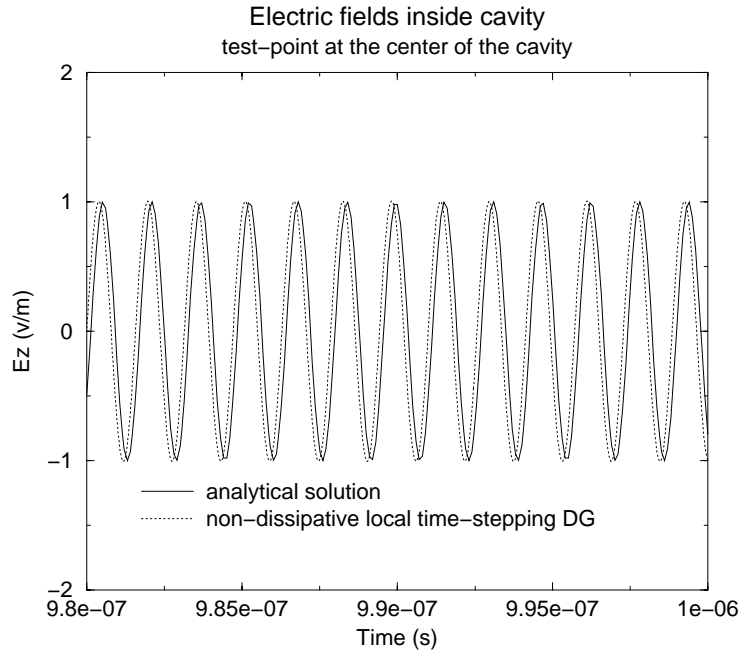


Fig. 8. Field evaluated at the center of a cavity by using the leap-frog multi-class local time-stepping method with a Q_3 non-dissipative spatial approach

recursive leap-frog local time-stepping strategy, and different FDTD solutions obtained by taking into account different spatial sizes of cells. The quantity under consideration is a scattered field inside the missile illuminated by a plane wave given by $k_y = 1$ and $E_x(t) = 377 * e^{-((t-2.e-10)/1.e-10)^2}$. We can

see the advantage to use the DG approach for accuracy and memory storage (reduction by 2 compared to FDTD). However, in terms of CPU-time, despite the use of a local time-stepping strategy for DG, the FDTD remains, in this example, more efficient (two times faster). This is not generally the case but in this example, this is mainly due to the very small size of a few cells in the unstructured mesh. In the future, we can improve this drawback by using in the DG method different spatial approximation order in each cell. In a such case, the smallest cells will have a low spatial order and a local time step larger and then the CPU-time will be reduced.

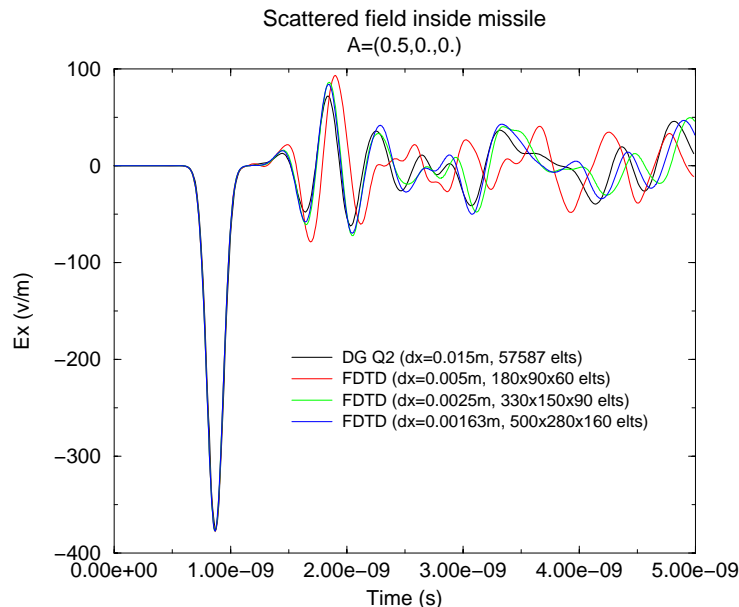


Fig. 9. Comparison of the solution obtained with FDTD and the proposed DG method using a spatial dissipative approach (dx in the DG label defines an average value)

5 Conclusion

In [8], a non-dissipative Discontinuous Galerkin method using a leap-frog time scheme has been developed to solve time domain Maxwell's equations. This scheme offers real advantages in terms of accuracy and memory storage compared to classical methods like FDTD for problems such as cavity or high frequency scattering problems. However, on industrial unstructured meshes in which very distorted and/or small cells appear, the DG method can suffer of important spurious modes and a too small time step. This generates a loss in the accuracy of the solution and a CPU-time considerably increased.

In this paper, we proposed and studied the introduction of dissipative penal-

ization terms and an original local time-stepping strategy. It has been proved mathematically and confirmed by means of examples that the dissipative terms provides a new formulation which improves the quality of the solution for distorted meshes and ensures the convergence of our DG scheme for all spatial approximation orders. In particular, we have also the convergence of the order 1 approximation, whereas it was not the case without these penalization terms. To accelerate the DG method, we propose also a local time-stepping strategy defined by a recursive multi-class method based upon a leap-frog scheme. Some comparisons with other approaches have been done and show the advantages of the proposed method. The statement of an explicit stability condition for this local time-stepping strategy is still an open problem, but numerical examples validated this method and shown that it can be applied without too many restrictions, even for cavity problems. In the future, the local time-stepping methods stay an important stake for Maxwell's equations and other ways [19] than the methods proposed here could be interesting to investigate.

Other improvements in terms of CPU-time and memory storage of the DG method proposed here will be studied in further works. In particular, considering different spatial approximation orders for each cells and using local time-stepping strategy will imply an additional reduction in CPU-time and memory storage.

A Appendix

Before giving the proofs of propositions (2) and (3), we assume some background relative to the hexahedral mesh considered here [10]:

- We assume that all hexahedron K are convex and, to characterize an element $K \in \mathcal{T}$, we define the diameter h_K of K and a regularity parameter $\sigma_K = \frac{h_K}{\rho_K}$ where $\rho_K = \|J_{F_K^{-1}}\|_{\infty, K}^{\frac{1}{3}}$ with $J_{F_K^{-1}}$ the determinant of the Jacobian matrix F_K^{-1} .
- We will use the estimates:

$$\begin{aligned} \|F_K\|_{1, \infty, \hat{K}} &\leq Ch_K \text{ and } \|J_K\|_{\infty, \hat{K}} \leq Ch_K^3 \\ \|F_K^{-1}\|_{1, \infty, K} &\leq C \frac{h_K^2}{\rho_K^3} \text{ and } \|J_{F_K^{-1}}\|_{\infty, K} = \rho_K^{-3} \\ \lambda((DF_K DF_K^*)(\hat{x})) &\leq Ch_K^2 \text{ and } \lambda((DF_K^{-1} DF_K^{*-1})(\hat{x})) \leq C \frac{h_K^4}{\rho_K^6}, \end{aligned} \quad (\text{A.1})$$

where $\lambda(A)$ belongs to the spectrum of A and $C > 0$ is independent of K .

- Finally, we consider a regular family $(\mathcal{T}_h)_{h>0}$ of triangulation of Ω such that when h tends toward 0, it exists a number $\sigma_c > 0$, independent of h , verifying $\forall K \in \mathcal{T} \sigma_K \leq \sigma_c$.

For the Maxwell equations, we assume also that $\exists \varepsilon_0, \mu_0 > 0$ such that $\forall x \in \Omega$, we have $\varepsilon(x) \geq \varepsilon_0$ and $\mu(x) \geq \mu_0$. In our analysis, the electric conductivity is $\sigma = 0$.

Proof of the proposition (2): $\forall \psi_h \in \mathbf{V}_r$, the exact solution of the Maxwell equations verifies in particular:

$$\int_K \varepsilon \frac{\partial E}{\partial t} \cdot \psi_h dx - \int_K \nabla \times H \cdot \psi_h dx = 0, \quad (\text{A.2})$$

and the approximate solution (E_h, H_h) :

$$\int_K \varepsilon \frac{\partial}{\partial t} E_h \cdot \psi_h dx - \int_K \nabla \times H_h \cdot \psi_h dx = \int_{\partial K} \left(\beta [H_h \times n]_{\partial K}^K + \lambda [n \times (E_h \times n)]_{\partial K}^K \right) \cdot \psi_h ds. \quad (\text{A.3})$$

By combining these equations (A.2) and (A.3) and by introducing elements $v_h, w_h \in \mathbf{V}_r$ as $E - E_h = E - v_h + v_h - E_h$ and $H - H_h = H - w_h + w_h - H_h$,

we obtain:

$$\begin{aligned}
& \int_K \varepsilon \frac{\partial}{\partial t} (E - v_h) \cdot \psi_h dx + \int_K \varepsilon \frac{\partial}{\partial t} (v_h - E_h) \cdot \psi_h dx \\
& - \int_K \nabla \times (H - w_h) \cdot \psi_h dx - \int_K \nabla \times (w_h - H_h) \cdot \psi_h dx \\
& = \int_{\partial K} \left(\beta \llbracket (H_h - w_h) \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times ((E_h - v_h) \times n) \rrbracket_{\partial K}^K \right) \cdot \psi_h ds \\
& + \int_{\partial K} \left(\beta \llbracket (w_h - H) \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times ((v_h - E) \times n) \rrbracket_{\partial K}^K \right) \cdot \psi_h ds,
\end{aligned} \tag{A.4}$$

To obtain the last line of the last expression, we used the fact that $\forall t, (E, H)(\cdot, t) \in H_0(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ ([17]) (ie $\forall \Gamma = K \cap K'$, one has $\llbracket H \times n \rrbracket_{\Gamma}^{K \text{ or } K'} = \llbracket E \times n \rrbracket_{\Gamma}^{K \text{ or } K'} = 0$) and that $\forall \Gamma = K \cap \partial\Omega$, $\beta = 0$ and $E \times n = 0$.

In a same way, one has:

$$\begin{aligned}
& \int_K \mu \frac{\partial}{\partial t} (H - w_h) \cdot \phi_h dx + \int_K \mu \frac{\partial}{\partial t} (w_h - H_h) \cdot \phi_h dx \\
& + \int_K \nabla \times (E - v_h) \cdot \phi_h dx + \int_K \nabla \times (v_h - E_h) \cdot \phi_h dx \\
& = \int_{\partial K} \left(\gamma \llbracket (E_h - v_h) \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times ((H_h - w_h) \times n) \rrbracket_{\partial K}^K \right) \cdot \phi_h ds \\
& + \int_{\partial K} \left(\gamma \llbracket (v_h - E) \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times ((w_h - H) \times n) \rrbracket_{\partial K}^K \right) \cdot \phi_h ds.
\end{aligned} \tag{A.5}$$

If one denotes $\Delta_E^P = E - v_h$, $\Delta_E^I = E_h - v_h$, $\Delta_H^P = H - w_h$, $\Delta_H^I = H_h - w_h$ and if one takes $\psi_h = \Delta_E^I$ and $\phi_h = \Delta_H^I$, one obtains:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{K \in \mathcal{T}} \int_K (\varepsilon \Delta_E^I \cdot \Delta_E^I + \mu \Delta_H^I \cdot \Delta_H^I) dx \\
& \leq \sum_{K \in \mathcal{T}} \left(\int_K (\varepsilon \frac{\partial}{\partial t} \Delta_E^P \cdot \Delta_E^I + \mu \frac{\partial}{\partial t} \Delta_H^P \cdot \Delta_H^I + \nabla \times \Delta_E^P \cdot \Delta_H^I - \nabla \times \Delta_H^P \cdot \Delta_E^I) dx \right) \\
& + \sum_{K \in \mathcal{T}} \int_{\partial K} \left(\beta \llbracket \Delta_H^P \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (\Delta_E^P \times n) \rrbracket_{\partial K}^K \right) \cdot \Delta_E^I ds \\
& + \sum_{K \in \mathcal{T}} \int_{\partial K} \left(\gamma \llbracket \Delta_E^P \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (\Delta_H^P \times n) \rrbracket_{\partial K}^K \right) \cdot \Delta_H^I ds.
\end{aligned} \tag{A.6}$$

In the evaluation of the last expression, the choice of β and γ in the DG formalism implies:

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \left(\int_K (\nabla \times \Delta_H^I \cdot \Delta_E^I - \nabla \times \Delta_E^I \cdot \Delta_H^I) dx - \int_{\partial K} \beta \llbracket \Delta_H^I \times n \rrbracket_{\partial K}^K \cdot \Delta_E^I ds \right. \\
& \quad \left. - \int_{\partial K} \gamma \llbracket \Delta_E^I \times n \rrbracket_{\partial K}^K \cdot \Delta_H^I ds \right) = 0,
\end{aligned} \tag{A.7}$$

and the choice of $\lambda > 0$ implies :

$$\sum_{K \in \mathcal{T}} \left(- \int_{\partial K} \lambda [n \times (\Delta_E^I \times n)]_{\partial K}^K \cdot \Delta_E^I ds - \int_{\partial K} \lambda [n \times (\Delta_H^I \times n)]_{\partial K}^K \cdot \Delta_H^I ds \right) \leq 0. \quad (\text{A.8})$$

We now consider $(v, w) \in \mathbf{V}_r \times \mathbf{V}_r$ such that $\forall (v', w') \in \mathbf{V}_r \times \mathbf{V}_r$ and $\forall K \in \mathcal{T}$,

$$\int_K \varepsilon v \cdot v' dx + \int_K \nabla \times w \cdot v' dx - \int_{\partial K} \left(\beta [w \times n]_{\partial K}^K + \lambda [n \times (v \times n)]_{\partial K}^K \right) \cdot v' ds = l_1(v')$$

$$\int_K \mu w \cdot w' dx - \int_K \nabla \times v \cdot w' dx - \int_{\partial K} \left(\gamma [v \times n]_{\partial K}^K + \lambda [n \times (w \times n)]_{\partial K}^K \right) \cdot w' ds = l_2(w'), \quad (\text{A.9})$$

where l_1, l_2 are the two linear forms on \mathbf{V}_r defined by: $l_1(v') = \int_K \varepsilon E \cdot v' dx + \int_K \nabla \times H \cdot v' dx$ and $l_2(w') = \int_K \mu H \cdot w' dx - \int_K \nabla \times E \cdot w' dx$.

Now, if we choose $v_h = v$ and $w_h = w$ solutions of (A.9), and by taking $v' = \Delta_E^I$ and $w' = \Delta_H^I$, we have :

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_K (\varepsilon \Delta_E^P \cdot \Delta_E^I + \mu \Delta_H^P \cdot \Delta_H^I) dx &= \sum_{K \in \mathcal{T}} \int_K (\nabla \times \Delta_E^P \cdot \Delta_H^I - \nabla \times \Delta_H^P \cdot \Delta_E^I) dx \\ &+ \sum_{K \in \mathcal{T}} \int_{\partial K} \left(\beta [\Delta_H^P \times n]_{\partial K}^K + \lambda [n \times (\Delta_E^P \times n)]_{\partial K}^K \right) \cdot \Delta_E^I ds \\ &+ \sum_{K \in \mathcal{T}} \int_{\partial K} \left(\gamma [\Delta_E^P \times n]_{\partial K}^K + \lambda [n \times (\Delta_H^P \times n)]_{\partial K}^K \right) \cdot \Delta_H^I ds \end{aligned} \quad (\text{A.10})$$

and (A.6) becomes:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{K \in \mathcal{T}} \int_K (\varepsilon \Delta_E^I \cdot \Delta_E^I + \mu \Delta_H^I \cdot \Delta_H^I) dx \\ &\leq \sum_{K \in \mathcal{T}} \int_K \left(\varepsilon \frac{\partial}{\partial t} \Delta_E^P \cdot \Delta_E^I + \mu \frac{\partial}{\partial t} \Delta_H^P \cdot \Delta_H^I \right) dx + \sum_{K \in \mathcal{T}} \int_K (\varepsilon \Delta_E^P \cdot \Delta_E^I + \mu \Delta_H^P \cdot \Delta_H^I) dx. \end{aligned} \quad (\text{A.11})$$

Then, by using Cauchy-Schwartz inequality and the inequalities $\frac{\|\Delta_E^I\|_{0,\varepsilon,\Omega}}{\|(\Delta_E^I, \Delta_H^I)\|_*} \leq 1$ and $\frac{\|\Delta_H^I\|_{0,\mu,\Omega}}{\|(\Delta_E^I, \Delta_H^I)\|_*} \leq 1$, (A.11) leads to:

$$\frac{d}{dt} \|(\Delta_E^I, \Delta_H^I)\|_* \leq \|\Delta_{\frac{\partial E}{\partial t}}^P\|_{0,\varepsilon,\Omega} + \|\Delta_{\frac{\partial H}{\partial t}}^P\|_{0,\mu,\Omega} + \|\Delta_E^P\|_{0,\varepsilon,\Omega} + \|\Delta_H^P\|_{0,\mu,\Omega} \quad (\text{A.12})$$

which concludes the proof.

Proof of the proposition (3) For this study, we follow the demonstration of [18] but in the non-affine case. In the following, we assume also that C, \tilde{C}, C_1

... define generic positive constants which are independent of $h = \max_{K \in \mathcal{T}}(h_K)$.

Let $a : \mathbf{V}_r^2 \times \mathbf{V}_r^2 \rightarrow \mathbb{R}$ be the bilinear form defined by:

$$\begin{aligned} a((u, v), (u', v')) &= \sum_{K \in \mathcal{T}} \left(\int_K \varepsilon u \cdot u' dx + \int_K \nabla \times v \cdot u' dx - \int_{\partial K} \left(\beta \llbracket v \times n \rrbracket_{\partial K}^K + \right. \right. \\ &\quad \left. \left. \lambda \llbracket n \times (u \times n) \rrbracket_{\partial K}^K \right) \cdot u' ds + \int_K \mu v \cdot v' dx - \int_K \nabla \times u \cdot v' dx \right. \\ &\quad \left. - \int_{\partial K} \left(\gamma \llbracket u \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (v \times n) \rrbracket_{\partial K}^K \right) \cdot v' ds \right). \end{aligned} \quad (\text{A.13})$$

To prove the proposition (3), we need essentially to show the continuity and a *inf-sup* property of the bilinear form a . First, we are going to give a *inf-sup* property by proving the following result:

Theorem 2 *There exists $C > 0$ (independent of h) such that*

$$\inf_{(u, v) \in \mathbf{V}_r \times \mathbf{V}_r} \sup_{(u', v') \in \mathbf{V}_r \times \mathbf{V}_r} \frac{a((u, v), (u', v'))}{\|(u, v)\|_h \|(u', v')\|_h} \geq C > 0, \quad (\text{A.14})$$

where

$$\begin{aligned} \|(u, v)\|_h^2 &= \|u\|_{0, \varepsilon, \Omega}^2 + \|v\|_{0, \mu, \Omega}^2 + \|\llbracket u \times n \rrbracket\|_{0, \mathcal{F}_h}^2 + \|\llbracket v \times n \rrbracket\|_{0, \mathcal{F}_h}^2 \\ &\quad + \sum_{K \in \mathcal{T}} h_K \left(\|\nabla \times u\|_{0, K}^2 + \|\nabla \times v\|_{0, K}^2 \right) \end{aligned}$$

and where \mathcal{F}_h denotes the set of faces of \mathcal{T} and

$$\|\llbracket u \times n \rrbracket\|_{0, \mathcal{F}_h}^2 = \sum_{\Gamma = K \cap K'} \int_{\Gamma} \|u_{K'} \times n_K - u_K \times n_K\|^2 ds + \sum_{\Gamma = K \cap \partial \Omega} \int_{\Gamma} \|u_K \times n_K\|^2 ds.$$

To prove this result, we begin to demonstrate three propositions on the majorations of the different terms of the bilinear form a . We choose test-functions u' and v' such that $u' \circ F_K = DF_K^{*-1} \hat{\nabla} \times \hat{v} = DF_K^{*-1} \hat{u}'$ and $v' \circ F_K = -DF_K^{*-1} \hat{\nabla} \times \hat{u} = DF_K^{*-1} \hat{v}'$. The first proposition gives a majoration of the terms of order 1 in the bilinear form a :

Proposition 4 *It exists $C > 0$ (independent of h) such that:*

$$\|u'\|_{0, \Omega} \leq C \|v\|_{0, \Omega} \text{ and } \|v'\|_{0, \Omega} \leq C \|u\|_{0, \Omega}. \quad (\text{A.15})$$

Proof of the proposition 4:

$$\begin{aligned}
\|u'\|_{0,K} &= \int_{\hat{K}} |J_K| u' \circ F_K \cdot u' \circ F_K d\hat{x} = \int_{\hat{K}} |J_K| DF_K^{*-1} \hat{\nabla} \times \hat{v} \cdot DF_K^{*-1} \hat{\nabla} \times \hat{v} d\hat{x} \\
&\leq Ch_K \int_{\hat{K}} \hat{\nabla} \times \hat{v} \cdot \hat{\nabla} \times \hat{v} d\hat{x} \leq C' h_K \int_{\hat{K}} \hat{v} \cdot \hat{v} d\hat{x} \text{ since } \hat{v} \in [Q_r(\hat{K})]^3 \\
&\leq C' h_K \int_{\hat{K}} \frac{|J_K|}{|J_K|} (DF_K^* DF_K^{*-1}) \hat{v} \cdot (DF_K^* DF_K^{*-1}) \hat{v} d\hat{x} \leq C'' \|v\|_{0,K}.
\end{aligned}$$

We have used (A.1) to obtain these estimates. The second estimate of the proposition is obtained by the same process.

The second proposition concerns the minoration of curl-terms in the bilinear form a :

Proposition 5 *It exists $C > 0$ (independent of h) such that:*

$$\begin{aligned}
-(\nabla_h \times u, v')_{0,\Omega} &= - \sum_{K \in \mathcal{T}} \int_K \nabla \times u \cdot v' dx \geq C \sum_{K \in \mathcal{T}} h_K \|\nabla \times u\|_{0,K}^2 \\
(\nabla_h \times v, u')_{0,\Omega} &= \sum_{K \in \mathcal{T}} \int_K \nabla \times v \cdot u' dx \geq C \sum_{K \in \mathcal{T}} h_K \|\nabla \times v\|_{0,K}^2.
\end{aligned}$$

Proof of the proposition 5:

$$\begin{aligned}
-(\nabla_h \times u, v')_{0,\Omega} &= - \sum_{K \in \mathcal{T}_h} (\nabla \times u, v')_{0,K} = - \sum_{K \in \mathcal{T}} \int_{\hat{K}} \hat{\nabla} \times \hat{u}_K \cdot \hat{v}'_K d\hat{x} \\
&= \sum_{K \in \mathcal{T}} \int_{\hat{K}} \hat{\nabla} \times \hat{u}_K \cdot \hat{\nabla} \times \hat{u}_K d\hat{x} \\
&= \sum_{K \in \mathcal{T}} \int_{\hat{K}} \frac{|J_K|}{|J_K|} \left(J_K DF_K^{*-1} \frac{DF_K^*}{J_K} \right) \hat{\nabla} \times \hat{u}_K \cdot \left(J_K DF_K^{*-1} \frac{DF_K^*}{J_K} \right) \hat{\nabla} \times \hat{u}_K d\hat{x} \\
&\geq C \sum_{K \in \mathcal{T}} h_K \|\nabla \times u\|_{0,K}^2.
\end{aligned}$$

We have used (A.1) to obtain these estimates. The second estimate of the proposition is obtained by the same process.

The third proposition concerns the jump terms of the bilinear form a :

Proposition 6 *It exists $C > 0$ (independent of h) such for all*

$\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}^*$, we have:

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_{\partial K} \beta [v \times n]_{\partial K}^K \cdot u' ds &\leq C \left(\delta_1^2 \| [v \times n] \|_{0, \mathcal{F}_h^i}^2 + \frac{1}{\delta_1^2} \sum_{K \in \mathcal{T}} h_K \| \nabla \times v \|_{0, K}^2 \right) \\ \sum_{K \in \mathcal{T}} \int_{\partial K} \lambda [n \times (u \times n)]_{\partial K}^K \cdot u' ds &\leq C \left(\delta_2^2 \| [u \times n] \|_{0, \mathcal{F}_h}^2 + \frac{1}{\delta_2^2} \sum_{K \in \mathcal{T}_h} h_K \| \nabla \times v \|_{0, K}^2 \right) \\ \sum_{K \in \mathcal{T}} \int_{\partial K} \lambda [n \times (v \times n)]_{\partial K}^K \cdot v' &\leq C \left(\delta_3^2 \| [v \times n] \|_{0, \mathcal{F}_h}^2 + \frac{1}{\delta_3^2} \sum_{K \in \mathcal{T}_h} h_K \| \nabla \times u \|_{0, K}^2 \right) \\ \sum_{K \in \mathcal{T}} \int_{\partial K} \gamma [u \times n]_{\partial K}^K \cdot v' ds &\leq C \left(\delta_4^2 \| [u \times n] \|_{0, \mathcal{F}_h}^2 + \frac{1}{\delta_4^2} \sum_{K \in \mathcal{T}} h_K \| \nabla \times u \|_{0, K}^2 \right). \end{aligned}$$

Proof of the proposition 6 We only prove the first inequality. One obtains the others by using the same technique. Using the Cauchy-Schwarz inequality, we can write:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \beta [v \times n]_{\partial K}^K \cdot u' ds \leq \sum_{K \in \mathcal{T}} \| \beta [v \times n]_{\partial K}^K \|_{0, \partial K} \| u' \|_{0, \partial K} \quad (\text{A.16})$$

Let $\delta_1 \neq 0$. By using $(\delta_1 a - \frac{b}{\delta_1})^2 = \delta_1^2 a^2 + \frac{b^2}{\delta_1^2} - 2ab \geq 0$ ($a, b \in \mathbb{R}$), (A.16) leads to:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \beta [v \times n]_{\partial K}^K \cdot u' ds \leq \sum_{K \in \mathcal{T}} \left(\frac{\delta_1^2}{2} \| \beta [v \times n]_{\partial K}^K \|_{0, \partial K}^2 + \frac{1}{2\delta_1^2} \| u' \|_{0, \partial K}^2 \right). \quad (\text{A.17})$$

We have also [10] $\forall u \in \mathbf{V}_r$ and $K \in \mathcal{T}$, it exists $C > 0$ (independent of K) such that $\| u|_K \|_{0, \partial K} \leq \frac{C}{h_K} \| u \|_{0, K}$. So, (A.17) becomes:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \beta [v \times n]_{\partial K}^K \cdot u' ds \leq \sum_{K \in \mathcal{T}} \left(\frac{\delta_1^2}{2} \| \beta [v \times n]_{\partial K}^K \|_{0, \partial K}^2 + \frac{C}{2\delta_1^2 h_K} \| u' \|_{0, K}^2 \right). \quad (\text{A.18})$$

Finally, the definition of u' gives $\| u' \|_{0, K}^2 \leq C h_K^2 \| \nabla \times v \|_{0, K}^2$ (for a certain constant $C > 0$) and then, by using (A.18), we complete the proof.

By using the proposition (5), we have:

$$\begin{aligned} a((u, v), (u', v')) &- \sum_{K \in \mathcal{T}} \left(\int_K (\varepsilon u \cdot u' + \mu v \cdot v') dx + \right. \\ &\quad \int_{\partial K} (\beta [v \times n]_{\partial K}^K + \lambda [n \times (u \times n)]_{\partial K}^K) \cdot u' ds \\ &\quad \left. + \int_{\partial K} (\gamma [u \times n]_{\partial K}^K + \lambda [n \times (v \times n)]_{\partial K}^K) \cdot v' ds \right) \\ &= \sum_{K \in \mathcal{T}} \int_K (-\nabla \times u \cdot v' + \nabla \times v \cdot u') dx \\ &\geq C \sum_{K \in \mathcal{T}} h_K (\| \nabla \times v \|_{0, K}^2 + \| \nabla \times u \|_{0, K}^2). \end{aligned}$$

By using the proposition (4), it exists a constant $C > 0$ such that

$$\int_K (\varepsilon u \cdot u' + \mu v \cdot v') dx \leq C(\|u\|_{0,\varepsilon,\Omega}^2 + \|v\|_{0,\mu,\Omega}^2). \quad (\text{A.19})$$

By using the proposition (6), we find a constant $C > 0$ such that for all $\delta_1, \delta_2, \delta_3, \delta_4 \neq 0$, we have:

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \left(\int_{\partial K} (\beta \llbracket v \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (u \times n) \rrbracket_{\partial K}^K) \cdot u' ds \right. \\ & \quad \left. + \int_{\partial K} (\gamma \llbracket u \times n \rrbracket_{\partial K}^K + \lambda \llbracket n \times (v \times n) \rrbracket_{\partial K}^K) \cdot v' ds \right) \\ & \leq C(\delta_1^2 + \delta_3^2) \|\llbracket v \times n \rrbracket\|_{0,\mathcal{F}_h}^2 + C(\delta_2^2 + \delta_4^2) \|\llbracket u \times n \rrbracket\|_{0,\mathcal{F}_h}^2 \\ & + C\left(\frac{1}{\delta_1^2} + \frac{1}{\delta_2^2}\right) \sum_{K \in \mathcal{T}} h_K \|\nabla \times v\|_{0,K}^2 + C\left(\frac{1}{\delta_3^2} + \frac{1}{\delta_4^2}\right) \sum_{K \in \mathcal{T}} h_K \|\nabla \times u\|_{0,K}^2. \end{aligned} \quad (\text{A.20})$$

Finally, by taking into account (A.19) and (A.20), one can always choose $\delta_1, \delta_2, \delta_3, \delta_4$ such that: $\exists C, \tilde{C} > 0$

$$\begin{aligned} & C \sum_{K \in \mathcal{T}} h_K (\|\nabla \times v\|_{0,K}^2 + \|\nabla \times u\|_{0,K}^2) \leq a((u, v), (u', v')) + \\ & \tilde{C} (\|u\|_{0,\varepsilon,\Omega}^2 + \|v\|_{0,\mu,\Omega}^2 + \|\llbracket u \times n \rrbracket\|_{0,\mathcal{F}_h}^2 + \|\llbracket v \times n \rrbracket\|_{0,\mathcal{F}_h}^2) \\ & = a((u, v), (u', v')) + \tilde{C} a((u, v), (u, v)). \end{aligned} \quad (\text{A.21})$$

By adding the term $C(\|u\|_{0,\varepsilon,\Omega}^2 + \|v\|_{0,\mu,\Omega}^2 + \|\llbracket u \times n \rrbracket\|_{0,\mathcal{F}_h}^2 + \|\llbracket v \times n \rrbracket\|_{0,\mathcal{F}_h}^2)$ to the previous inequality, we obtain:

$$C\|(u, v)\|_h^2 \leq a((u, v), (u' + C_1 u, v' + C_1 v)), \quad (\text{A.22})$$

where $C_1 = C + \tilde{C}$.

Now, we are going to prove that $\|(u', v')\|_h \leq C\|(u, v)\|_h$. By using proposition (4), we obtain:

$$\|u'\|_{0,\varepsilon,\Omega} \leq C\|v\|_{0,\mu,\Omega} \text{ and } \|v'\|_{0,\mu,\Omega} \leq C\|u\|_{0,\varepsilon,\Omega}.$$

By using (A.1), we have:

$$\begin{aligned} \|\nabla \times u'\|_{0,K}^2 & \leq \frac{C}{h_K} \|\hat{\nabla} \times \hat{u}'\|_{0,\hat{K}}^2 \leq \frac{C'}{h_K} \|\hat{u}'\|_{0,\hat{K}}^2 \text{ because } u' \in [Q_r(\hat{K})]^3 \\ & \leq \frac{C'}{h_K} \|\hat{\nabla} \times \hat{v}\|_{0,\hat{K}}^2 \leq C'' \|\nabla \times v\|_{0,K}^2 \end{aligned}$$

and

$$\|\nabla \times v'\|_{0,K}^2 \leq C\|\nabla \times u\|_{0,K}^2.$$

Moreover, we can prove:

$$\|\llbracket u' \times n \rrbracket\|_{0,\mathcal{F}_h}^2 \leq C \sum_{K \in \mathcal{T}} h_K \|\nabla \times v\|_{0,K}^2$$

and

$$\| [v' \times n] \|_{0, \mathcal{F}_h}^2 \leq C \sum_{K \in \mathcal{T}} h_K \|\nabla \times u\|_{0, K}.$$

Indeed, it is easy to see that

$$\| [u' \times n] \|_{0, \mathcal{F}_h}^2 \leq \sum_{K \in \mathcal{T}} \int_{\partial K} \|u'_K \times n_K\|^2 ds \leq \sum_{K \in \mathcal{T}} \int_{\partial K} \|u'_K\|^2 ds \quad (\text{A.23})$$

Now, by using (A.1), it exists a constant C such that:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \|u'_K\|^2 ds \leq C \sum_{K \in \mathcal{T}} h_K \int_K \nabla \times v \cdot \nabla \times v dx$$

which gives the expected result. Finally, if we take $(w_1, w_2) = (u' + C_1 u, v' + C_1 v) \in \mathbf{V}_r \times \mathbf{V}_r$, (A.22) leads to

$$C \| (u, v) \|_h \| (w_1, w_2) \|_h \leq a((u, v), (w_1, w_2)), \quad \forall (u, v) \in \mathbf{V}_r \times \mathbf{V}_r$$

for a certain constant $C > 0$ independent of h . So, we obtain the *inf-sup* condition (A.14).

The next step of the demonstration of the proposition (3) is to verify the continuity of the bilinear form a . To obtain this result, you have the following lemma:

Lemma 1 *It exists $M > 0$ (independent of h) such that*

$$a((u, v), (u', v')) \leq M \| (u, v) \|_{h, \frac{1}{2}} \| (u', v') \|_h, \quad \forall u, v, u', v' \in \mathbf{V}_r$$

where $\| (u, v) \|_{h, \frac{1}{2}}^2 = \| (u, v) \|_h^2 + \sum_{K \in \mathcal{T}} \left(h_K^{-1} (\|u\|_{0, K}^2 + \|v\|_{0, K}^2) + \|u\|_{0, \partial K}^2 + \|v\|_{0, \partial K}^2 \right)$.

To verify this lemma, it is sufficient to integrate by parts $a((u, v), (u', v'))$ (For more detail see [18]).

Recall that we solve the problem: find $(u, v) \in \mathbf{V}_r \times \mathbf{V}_r$ such that $\forall (u', v') \in \mathbf{V}_r \times \mathbf{V}_r$, $a((u, v), (u', v')) = (l_1(u'), l_2(v'))$. So, we have immediately the consistency result : $a((u - E, v - H), (u', v')) = 0$ for all (u', v') in $\mathbf{V}_r \times \mathbf{V}_r$. Now the *inf-sup* condition and the continuity of the bilinear form lead to the well-known estimate:

$$\| (u - E, v - H) \|_h \leq C \inf_{(u', v') \in \mathbf{V}_r \times \mathbf{V}_r} \| (E - u', H - v') \|_{h, \frac{1}{2}}$$

for a certain constant $C > 0$. In [10], we have derived for a particular projector π_h the following interpolations error: let $v \in [H^{s+1}(K)]^3$ with $s \geq 0$ a real.

There exists a constant $C > 0$ independent of h such that

$$\begin{aligned} \|v - \pi_{h|K} v\|_{0,K} &\leq Ch_K^{\min(s,r)} \|v\|_{s+1,K} \\ |v - \pi_{h|K} v|_{1,K} &\leq Ch_K^{\min(s-1,r-1)} \|v\|_{s+1,K} \\ \|(v - \pi_{h|K} v) \times n_K\|_{0,\partial K} &\leq Ch_K^{\min(s-\frac{1}{2},r-\frac{1}{2})} \|v\|_{s+1,K}. \end{aligned}$$

So, If we assume that the exact solution verifies $(E, H) \in H^{s+1}(\mathcal{T})$ for $s \geq 0$, then there exists a constant $C > 0$ such that:

$$\|u - E, v - H\|_* \leq \|(u - E, v - H)\|_h \leq Ch^{\min(s-\frac{1}{2},r-\frac{1}{2})} \max(\|E\|_{s+1,h}, \|H\|_{s+1,h})$$

where $H^s(\mathcal{T}) = \{v \in [L^2(\Omega)]^3 : \forall K \in \mathcal{T}, v|_K \in [H^s(K)]^3\}$ and $\|v\|_{s,h}^2 = \sum_{K \in \mathcal{T}} \|v\|_{s,K}^2$ and the proposition (3) is verified.

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