Desingularization of non local dynamic models by means of operatorial transformations and application to a flame model

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Abstract: In this paper, we show how suitable operatorial transformations can be used to simplify non local singular dynamic models. First, some basic notions on trajectories, operators and dynamic problems are introduced. Second, the diffusive transformation of non local models is briefly exposed. Then, time-scale transformations are introduced and some essentials properties are stated; the desingularization of models (in general non time local) is then studied. Finally, those transformations are applied on a singular implicit and non local flame model.

Keywords: Operatorial transformations, time-scale transformation, singular models, nonlocal models, diffusive transformation, flame model.

1. INTRODUCTION

Most of the time, dynamic problems are formulated in the state space, as differential equations. However, there are many problems that need to be formulated in a global manner, that is in trajectories spaces, e.g. integro-differential models, that need to be analyzed and solved in trajectories spaces. Such models are said "time non local" (differential models being on the contrary time-local).

Then, such formulations allow to consider larger classes of dynamic systems. Moreover, they also allow the use of more general transformations (e.g. global transformations, non necessarily local) of dynamic models and problems, and bring new possibilities of simplifications of dynamic problems such that analysis, simulation, identification, control, etc.

In this paper, we show how suitable operatorial transformations can be used to "desingularize" local as well as non local singular dynamic models, and then simplify the resolution of classical dynamic problems on it. First, some basic notions on trajectories, operators and dynamic problems are introduced. Second, the diffusive transformation of non local models is briefly exposed. Then, time-scale transformations are introduced and some essentials properties are stated; the desingularization of models (in general non time local) is then studied. Finally, those transformations are applied on a singular implicit and non local flame model Joulin [1985], whose analysis and numerical simulation are simplified.

2. TRAJECTORIES, DYNAMIC PROBLEMS, OPERATORIAL TRANSFORMATIONS

We introduce in this section some basic notions about trajectories and operators used in the sequel. We propose an abstract formulation of dynamic problems, well adapted to the problematic of operatorial transformations, whose aim is to give a global vision of the majority of dynamic problems classically encountered, and then allow to study generic operatorial transformations on it.

We call *trajectory* any function x defined on a time interval I (most of the time $[0,T]$) with values in a topological vectorial (usually Banach) space $\mathbf{X}$ (e.g. $L^\infty(\mathbb{R}_t;\mathbf{X})$, $C^0(\mathbb{R}_t;\mathbf{X})$, etc.). An *operator* is a function between two trajectories spaces.

Definition 1. An operator $\mathcal{H}:\mathfrak{X}\to\mathfrak{Y}$ is said:

- *Static* if $(\mathcal{H}(f))(t)=H(t,f(t))$ with H a classical function. In the sequel, for simplicity of notations, if H is a function we will also denote by H the associated static operator.
- *Causal* when $\forall u,v\in\operatorname{dom}(\mathcal{H})$, $\forall t\in\mathbb{R}$,
  $\operatorname{supp}(u-v)\subset[t,+\infty[\;\Rightarrow\operatorname{supp}(\mathcal{H}(u)-\mathcal{H}(v))\subset[t,+\infty[\,.$
- (time-)*local* if for any $t\in I$ and for any functions $u,v\in\mathfrak{X}$ such that $u=v$ on a neighborhood of t, we have:
  $$(\mathcal{H}u)(t)=(\mathcal{H}v)(t).$$

Definition 2. We call *model* an equation of the form:
$$\mathbf{\Phi}(u,x)=0,\ (u,x)\in\mathcal{U}\times\mathcal{X},\tag{1}$$
where $\mathcal{U},\mathcal{X}$ are topological manifolds, $\mathcal{S}$ a vectorial topological space an $\mathbf{\Phi}:\mathcal{U}\times\mathcal{X}\to\mathcal{S}$ an operator.

The model (1) is said *dynamic explicit* when $\mathcal{U},\mathcal{X}$ are manifolds of trajectories spaces and $\mathbf{\Phi}(u,x)=\mathcal{H}x-\mathcal{F}(u,x)$ where:

- $\mathcal{H}$ is a linear causal invertible operator,
- $\mathcal{F}(u,\cdot)$ is a static operator.

Finally, it is said *local* when $\mathbf{\Phi}$ is a local operator.

Example 1. We consider the explicit differential system in $\mathbb{R}^n$:

$$\begin{cases} \partial_t x = f(v,x) \\ x(0) = x_0; \end{cases}$$

then the system can be written $\mathbf{\Phi}(u,x) := \mathcal{H}x - \mathcal{F}(u,x) = 0$, with $u := (v, x_0)^T$ and:

$$\mathcal{H} = \begin{pmatrix} \partial_t \\ \delta \end{pmatrix}, \quad \mathcal{F}(u,x) = \begin{pmatrix} f(u,x) \\ x_0 \end{pmatrix}$$

with (for example): $\mathcal{X} \subset C^1([0,T];\mathbb{R}^n)$, $\mathcal{U} \subset C^0([0,T];\mathbb{R}^p) \times \mathbb{R}^n$, and where δ is the Dirac operator. This model is dynamic explicit and local.

Definition 3. We call *dynamic problem* any system of equations of the form:

$$\begin{cases} \mathbf{\Phi}(u,x) = 0 \\ \mathbb{P}(u,x) = 0, \end{cases} \quad (u,x) \in \mathcal{U} \times \mathcal{X}, \tag{2}$$

where $\mathbf{\Phi} = 0$ is a dynamic model and $\mathbb{P} : \mathcal{U} \times \mathcal{X} \to \{0,1\}$ is a "property [1]".

Namely, the property $\mathbb{P}$ could have the expression $\mathbf{J}(u,x) = 0$ with $\mathbf{J}$ a continuous operator, or $\mathcal{J}(u,x) = \min$ with $\mathcal{J}$ a cost functional. This global formulation includes the majority of classical dynamic problems encountered in practice, such as control, identification, estimation, analysis, numerical simulation etc. Montseny [2009].

Then, we call *transformation* any operator on trajectories spaces that transforms a problem $(\mathbf{\Phi}, \mathbb{P}) = 0$ on $\mathcal{U} \times \mathcal{X}$ into a "new" problem $(\tilde{\mathbf{\Phi}}, \tilde{\mathbb{P}}) = 0$ on $\tilde{\mathcal{U}} \times \tilde{\mathcal{X}}$, such that:

$$\tilde{\mathbf{\Phi}}(\tilde{u}, \tilde{x}) = 0 \Rightarrow \mathbf{\Phi}(u,x) = 0,$$

that is the solution of the initial problem is implied by the solution of the "new" problem.

3. DIFFUSIVE TRANSFORMATIONS OF NON LOCAL MODELS

3.1 Diffusive representation of causal convolution operators

We only give a few basic notions about diffusive representation theory. A complete statement of diffusive representation can be found in Montseny [2005]. Various applications of this approach will be found for example in Carmona and Coutin [1998], Degerli et al. [1999], Garcia and Bernussou [1998], Mouyon and Imbert [2002], Rumeau et al. [2006].

We consider a causal convolution operator defined, on any continuous function $u : \mathbb{R}^+ \to \mathbb{R}$, by

$$u \mapsto \int_0^t h(t-s)\, u(s)\, ds. \tag{3}$$

We denote H the Laplace transform of the (locally integrable) function h, and $H(\partial_t)$ the convolution operator defined by (3). We suppose [2] that $H(p) \underset{|p| \to +\infty}{\to} 0$. The *diffusive representation* allows to write a formulation of operator $H(\partial_t)$ by means of a state equation on $(t,\xi) \in \mathbb{R}^{*+} \times \mathbb{R}$ (of diffusive type), time-local, of the form:

$$\begin{cases} \partial_t \psi(t,\xi) = \gamma(\xi)\, \psi(t,\xi) + u(t), \;\; \psi(0,\xi) = 0 \\ (H(\partial_t)u)\,(t) = \displaystyle\int_{\mathbb{R}} \mu(t,\xi)\, \psi(t,\xi)\, d\xi, \end{cases} \tag{4}$$

where:

- γ is a function defining a close (possibly at infinity) arc in $\mathbb{C}^-$ including all singularities of the symbol $H(p)$ of $H(\partial_t)$.
- μ is a functional object (in general a distribution) depending on γ and fully summarizing the operator $H(\partial_t)$. If the singularities of H on γ are branching points such that $|H \circ \gamma|$ is locally integrable in their neighborhood, then we have the following expression of μ:

$$\mu = \frac{\gamma'}{2\mathrm{i}\pi}\, K \circ \gamma. \tag{5}$$

The partial function $\psi(t,.)$ is called the *diffusive representation* of u, denoted $(\mathfrak{R}\mathrm{d}_\gamma u)(t)$, and μ the *γ-symbol* of $H(\partial_t)$.

The system (4) is of infinite dimension (in ξ); however, its diffusive nature allows to build economic numerical realisations by using a few points of discretization in ξ.

3.2 Diffusive transformations of non local models

We consider the *non local* dynamic models of the form:

$$\mathbf{\Phi}(u,x) := \mathbf{\Psi}(u,x,H(\partial_t) f(u,x)) = 0, \tag{6}$$

where the operator $\mathbf{\Psi}$ is *static* and $K(\partial_t)$ a γ-diffusive operator of γ-symbol μ. By applying the operator $\mathfrak{R}\mathrm{d}_\gamma$ on $f(u,x)$ (i.e. using the diffusive representation of $f(u,x)$), this model can be written:

$$\mathbf{\Psi}(u,x,\mu \cdot \mathfrak{R}\mathrm{d}_\gamma f(u,x)) = 0,$$

where $\mathfrak{R}\mathrm{d}_\gamma f(u,x)$ is solution of $\partial_t \psi = \gamma\psi + f(u,x)$, $\psi(0,.) = 0$. As a consequence, (6) is transformed into a *local* explicit dynamic model, with state (x,ψ):

$$\mathbf{\Phi}(u,x,\psi) = 0,$$

with:

$$\mathbf{\Phi}(u,x,\psi) = \begin{pmatrix} \mathbf{\Psi}(u,x,\mu \cdot \psi) \\ \partial_t \psi - \gamma\psi - f(u,x) \\ \boldsymbol{\delta} \cdot \psi \end{pmatrix}.$$

Thus:

Theorem 4. Under the above mentioned hypothesis, the operator $\mathfrak{R}\mathrm{d}_\gamma$ transforms any non local dynamic model of the form (6) into a local dynamic model of augmented space.

An important case is *explicit* non local models, of the form:

$$H(\partial_t)x = F(u,x), \tag{7}$$

which becomes, when $H(\partial_t)^{-1}$ admits a γ-symbol μ:

$$x = H(\partial_t)^{-1} F(u,x) = \mu \cdot \mathfrak{R}\mathrm{d}_\gamma F(u,x),$$

that is:

$$\begin{cases} \partial_t \psi = \gamma\psi + F(u,x), \quad \psi(0,.) = 0 \\ x = \mu \cdot \psi. \end{cases} \tag{8}$$

Remark 1. With the hypothesis $\gamma(0) = 0$, we see that any differential model of the form:

$$\partial_t x = F(u,x) \tag{9}$$

[1] Property that would be satisfied by (u,x), with the convention: 0 ∼True and 1 ∼False.

[2] This condition can easily be relaxed, e.g. there exists $n \in \mathbb{N}^*$ such that:

$$p^{-n} H(p) \underset{|p| \to +\infty}{\to} 0.$$

is trivially under diffusive formulation (8) with $\mu = \delta$ (that is $x(t) = \psi(t, 0)$); then, from a formal point of view, the model (9) can be considered as a particular case of (8).

4. TIME-SCALE TRANSFORMATIONS

There exists many examples of dynamic systems which have a natural internal clock under which the dynamic equations are significatly simplified Fangtang and Kelkari [2003], Bliman and Sorine [1996]. A generic formulation of time-scale transformations (TST) and their study reveals several interesting properties. In the following, $\mathbf{X}$ is a Banach space and $\mathfrak{X}$ a space of trajectories in $\mathbf{X}$ (e.g. $L^\infty(I; \mathbf{X})$), and ∂_t and ∂_t^{-1} are respectively the derivative and integration operators.

4.1 Definitions and properties

Basically, a TST operator is an operator of the form $x \mapsto x \circ \varphi^{-1}$ where $\varphi(t)$ is a new time-scale, defined by an invertible [3] function φ.

Definition 5. The general operator of time-scale transformation $\mathbf{S}$ is defined, for any strictly increasing continuous function φ and any trajectory $x \in \mathfrak{X}$, by:

$$\mathbf{S}(x, \varphi) := x \circ \varphi^{-1}. \tag{10}$$

- By convenience, we denote:

$$\mathbf{S}_\varphi := \mathbf{S}(\cdot, \varphi).$$

- For simplicity, we denote $\tau := \varphi(t)$ the "new time" and $\widetilde{x}$ the trajectory x after time-scale transformation, that is:

$$\widetilde{x} := \mathbf{S}_\varphi(x) = x \circ \varphi^{-1}.$$

The following property is important from a topological point of view.

Proposition 6. A TST operator $\mathbf{S}_\varphi$ is isometric from $L^\infty(I; \mathbf{X})$ in $L^\infty(\varphi(I); \mathbf{X})$.

As a consequence, TST operators preserve stability of trajectories of a dynamic system.

The inversion of a time-scale transformation, essential to keep equivalence of models, is simple to get.

Proposition 7. The inverse of a TST operator $\mathbf{S}(\cdot, \varphi)$ is given by the operator $\mathbf{S}(\ \cdot\ , \varphi^{-1})$, that is:

$$\mathbf{S}_\varphi^{-1} = \mathbf{S}_{\varphi^{-1}}\ . \tag{11}$$

Moreover, we can state the following expressions of a TST operator and its inverse:

Proposition 8. If $\varphi \in C^1(I; \mathbb{R})$ with $\varphi' > 0$, then we have

$$\varphi^{-1} = \partial_\tau^{-1}\left(\frac{1}{\widetilde{\varphi'}}\right), \tag{12}$$

that gives:

$$\mathbf{S}_\varphi^{-1} = \mathbf{S}_{\partial_\tau^{-1}\frac{1}{\widetilde{\varphi'}}} \quad \text{and} \quad \mathbf{S}_\varphi = \mathbf{S}_{\left[\partial_\tau^{-1}\frac{1}{\widetilde{\varphi'}}\right]^{-1}}\ . \tag{13}$$

[3] However, the use of non invertible time-scale transformation is not excluded, namelly when dealing with models involving dry friction, see for example Montseny [2009].

Finally, we have the following results, relating to the composition of a TST operator and the derivative operator.

Proposition 9. Let $\varphi \in C^1(I; \mathbb{R})$ with $\varphi' > 0$ and x differentiable. Then, we have:

$$\mathbf{S}_\varphi \circ \partial = \mathbf{S}_\varphi(\varphi')\ (\partial \circ \mathbf{S}_\varphi)\,.$$

In a more condensed way:

$$\widetilde{x'} = \widetilde{\varphi'}\ \widetilde{x}\,'. \tag{14}$$

4.2 Dynamic time-scale transformations

In the definition of a time-scale transformation, φ is a given function, that can be the result of a (dynamic) transformation of a function v, that is $\varphi = \boldsymbol{\varphi}(v)$ with $\boldsymbol{\varphi}$ a *dynamic operator.*

Definition 10. A TST operator is said *dynamic* when $\varphi = \boldsymbol{\varphi}(v)$ with $\boldsymbol{\varphi}$ a dynamic operator defined on a trajectories manifold $\mathcal{V}$ such that $\forall v \in \mathcal{V}$, $\boldsymbol{\varphi}(v)$ is continuous and strictly increasing.

In this case, we denote $\mathbf{S}_{\boldsymbol{\varphi}}$ the operatorial function:

$$\mathbf{S}_{\boldsymbol{\varphi}} : v \mapsto \mathbf{S}_{\boldsymbol{\varphi}(v)}.$$

As the time t and $\tau = \varphi(t)$ are different, the causality of such operators cannot be defined in the classical way. However, we can define a notion of causality of such operators, important from their practicability and real-time implementation.

Definition 11. The TST operator $\mathbf{S}_{\boldsymbol{\varphi}}$ is said causal if and only if *the operator* $\boldsymbol{\varphi}$ is causal.

Then, with such TST, the clock $\boldsymbol{\varphi}(v)$ is determined from v in a causal way and can be concretely implemented in real-time.

Remark 2. A time-scale transformation can be applied on the trajectory v itself :

$$\widetilde{v} = \mathbf{S}_{\boldsymbol{\varphi}(v)}\,(v) = v \circ \boldsymbol{\varphi}(v)^{-1}.$$

If the operator $\boldsymbol{\varphi}$ is causal, this expression is simple to implement and compatible with real time applications.

An important example of causal dynamic TST operator is given by $\boldsymbol{\varphi} = \partial_t^{-1}$, that is the TST operator $\mathbf{S}_{\partial_t^{-1} v}$, for which we can deduce from proposition 8 the following relations:

$$\mathbf{S}_{\partial_t^{-1} v} = \mathbf{S}^{-1}_{\partial_\tau^{-1}(\frac{1}{\widetilde{v}})} = \mathbf{S}_{\left[\partial_t^{-1}(v)\right]^{-1}}. \tag{15}$$

Finally, we can state the following proposition, on which is based the singularity simplification detailed on the next paragraph.

Proposition 12. Let $g \in C^0(I; \mathbb{R}^{+*})$ and x differentiable. Then:

$$\mathbf{S}\left(g\, x'\,,\, \partial_t^{-1}\frac{1}{g}\right) = \left[\mathbf{S}\left(x,\, \partial_t^{-1}\frac{1}{g}\right)\right]'. \tag{16}$$

In a more condensed way:

$$g\, \partial_t \overset{\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}}{\rightsquigarrow} \partial_\tau.$$

Note that, for simplicity of notations, we denoted g in the above proposition a function of t, that can be of the form $G(t, u, x)$ with G a static operator.

4.3 Suppression of model singularities

Transformation of local model

Proposition 13. We consider the dynamic model:

$$\mathbf{\Phi}(u, x, g\,\partial_t x) = 0, \tag{17}$$

with $\mathbf{\Phi}$ a static operator and $g \in C^0(I; \mathbb{R}^{+*})$. Then, by the time-scale transformation $\mathbf{S}_{\partial_t^{-1}\frac{1}{g}}$, the equation (17) is transformed into:

$$\mathbf{\Phi}(\tilde{u}, \widetilde{x}, \partial_\tau \widetilde{x}) = 0, \tag{18}$$

the correspondence between τ and t being indifferently defined by $\partial_t \tau = \frac{1}{g}$ or $\partial_\tau t = \widetilde{g}$.

Then, a suitable TST can allow to transform a first order differential equation $g\,\partial_t x = F(u, x)$, whose differential operator is with variable coefficient g (and eventually depending on x), into the following differential equation:

$$\partial_\tau \widetilde{x} = F(\tilde{u}, \tilde{x}).$$

In other words, such transformation allow to suppress some undesirable terms of a model by "absorbing" them into the operator ∂_τ. Then, the resolution of dynamic problems posed on those models would be simplified.

In the case of non local models, the diffusive transformation allow to apply this result, which is for this reason generic for a various class of dynamic problems.

Desingularization of singular non local models We consider a non local model of the form [4] :

$$H(\partial_t)\,x = \frac{1}{h(t, u, x)}\,F(t, u, x), \tag{19}$$

where F and h are static operators, $H(\partial_t)^{-1}$ a diffusive linear dynamic operator, and $h(t, u, x) > 0$ sometimes "close to the value 0", making this model singular. A diffusive transformation of the model leads to the formulation:

$$\partial_t \psi = \gamma\,\psi + \frac{1}{h(t, u, x)}\,F(t, u, x),\ \ \psi(0, \xi) = 0 \tag{20a}$$

$$x = \mu \cdot \psi. \tag{20b}$$

In such a situation, the manipulation of realization (20a) is tricky because of the presence of the singular product $g(t, u, x)\,F(t, u, x)$. The equation (20a) can also be written:

$$h(t, u, x)\,\partial_t \psi = h(t, u, x)\gamma\,\psi + F(t, u, x),\ \ \psi(0, \xi) = 0.$$

Then, thanks to the proposition 13, the dynamic time-scale transformation:

$$\mathbf{S} : x \mapsto x \circ \left[\partial_t^{-1} \frac{1}{h(t, u, x)}\right]^{-1}, \tag{21}$$

leads to an equivalent model in time τ:

$$\partial_\tau \widetilde{\psi} = \widetilde{h}(\tau, \widetilde{u}, \widetilde{x})\gamma\widetilde{\psi} + \widetilde{F}(\tau, \widetilde{u}, \widetilde{x}),\ \ \widetilde{\psi}(0, \xi) = 0. \tag{22}$$

Thus:

Theorem 14. Using the TST (21), the singular model (20) is transformed into:

$$\begin{cases} \partial_\tau \widetilde{\psi} = \widetilde{h}(\tau, \widetilde{u}, \widetilde{x})\gamma\widetilde{\psi} + \widetilde{F}(\tau, \widetilde{u}, \widetilde{x}),\ \ \widetilde{\psi}(0, \xi) = 0 \\ \widetilde{x} = \mu \cdot \widetilde{\psi}\ , \end{cases} \tag{23}$$

the correspondence between t and τ being defined by $\partial_\tau t = \widetilde{h}(\tau, \widetilde{u}, \widetilde{x})$.

This regular formulation can be used to investigate classical analysis and numerical simulation techniques devoted to differential equations.

Remark 3. Model transformations by means of time-scale transformations suggests other applications than the suppression of singularities; we can mention the possibilities of the decoupling state/input in dynamic models of the form:

$$H(\partial_t)\,x = A(x) + B(u)\,C(x),$$

or transformations of differential inclusions into classical differential equations. One can refer to Montseny and Camon [2010], Montseny [2009], Montseny and Doncescu [2008] for some examples of application.

5. APPLICATION TO ANALYSIS AND SIMULATION OF A SINGULAR FLAME MODEL

In Joulin [1985], Joulin proposed a model for the evolution of a spherical flame, which presents many difficulties: it is non local, implicit and singular. Then, the analysis and the simulation of this model is tricky Audounet et al. [1998, 2002].

We show in this section how suitable operatorial transformations lead to natural and simpler formulations of the model, presenting interesting properties both for analysis and numerical simulations purposes, and eventually control (not treated here).

5.1 Model under consideration

The model proposed by Joulin is given by:

$$\begin{cases} \frac{1}{2}\,x\,\partial_t^{\frac{1}{2}} x = x \ln x + u, \quad t > 0 \\ x(0^+) = 0,\ u \geqslant 0,\ x \geqslant 0, \end{cases} \tag{24}$$

where x is the flame radius.

By diffusive transformation via the contour $\gamma(\xi) = -|\xi|$, we get the following model [5] :

$$\begin{cases} x\,\partial_t \psi(t, \xi) = -|\xi|\ \psi + x \ln(x) + u,\ \ \psi(0, \xi) = 0 \\ x = <\mu, \psi> = \displaystyle\int_{\mathbb{R}} \frac{1}{\pi\sqrt{|\xi|}}\,\psi(., \xi)\,d\xi. \end{cases} \tag{25}$$

Even if this time-local formulation is advantageous compared to (24), it is still singular at origin, what makes its analysis tricky and leads to difficulties for numerical simulations, because standard techniques cannot be investigated Audounet et al. [2002].

5.2 Time-scale transformation of the model

Following the results stated in section 4.3, using the dynamic TST:

$$\mathbf{S}_{\partial_t^{-1}\frac{1}{x}} : z \mapsto \widetilde{z} := v \circ \left(\partial_t^{-1} \frac{1}{x}\right)^{-1},$$

[4] Local model are of course included by this expression by considering $H(\partial_t) = \partial_t$.

[5] The γ-symbol μ of the operator $2\partial_t^{-\frac{1}{2}}$ belongs to $L^1_{\text{loc}}(\mathbb{R})$ and is given by Montseny [2005]:

$$\mu(\xi) = \frac{1}{\pi\sqrt{|\xi|}} \geqslant 0.$$

we get the following equivalent model:

$$\begin{cases} \partial_\tau \widetilde{\psi} = -\widetilde{x}\,|\xi|\,\widetilde{\psi} + \widetilde{x}\,\ln(\widetilde{x}) + \widetilde{u},\ \widetilde{\psi}(0,\xi) = 0 \\ \widetilde{x} = \displaystyle\int_{\mathbb{R}} \frac{1}{\pi\sqrt{|\xi|}}\,\widetilde{\psi}(\cdot,\xi)\,d\xi, \end{cases} \tag{26}$$

the correspondence between t and τ being defined by:

$$\partial_\tau t = \widetilde{x}. \tag{27}$$

The model (26) is no longer singular (but still strongly nonlinear). The time τ reveals itself to be the "natural time" of the system. Indeed, the relation (27) shows that the smaller the flame radius $\widetilde{x}$, the faster the evolution of time τ towards time t; in other words, when close to the singularity $\widetilde{x} = 0$, a time-step $\Delta\tau$ represents a small step Δt, and vice versa when $\widetilde{x}$ is larger. Finally, the time τ "follows" the system evolution in the sense that the time $t(\tau)$ naturally stops when the flame vanishes. The formulation (26) allowed to realize precise numerical simulations to validate an identification method introduced in Casenave [2009, 2010], whereas simulation of the singular model induced systematic identification error close to 0.

5.3 Analysis of dissipativity of the model

The formulation (26) is well adapted to analysis of the model using classical energy methods. In particular, we consider the following energy functional, defined for any solution φ of (26):

$$E_\varphi := \frac{1}{2}\int_{\mathbb{R}} \mu\,\varphi^2\,d\xi \geqslant 0.$$

Then, simple computations lead to the following result.

Proposition 15. For any solution φ of (26), we have:

$$\frac{d}{d\tau}E_\varphi(\tau) = -\int_{\mathbb{R}} \widetilde{x}\,|\xi|\,\mu\,\varphi^2\,d\xi - \widetilde{x}^{\,2}\ln\widetilde{x} + \widetilde{u}\,\widetilde{x};$$

in particular, if $u = 0$, we have:

$$\frac{d}{d\tau}E_\varphi(\tau) \leqslant 0. \tag{28}$$

Then, we have the energy balance:

$$E_\varphi(\tau) = -\int_0^\tau \left[\int_{\mathbb{R}} \widetilde{x}\,|\xi|\,\mu\,\varphi^2\,d\xi - \widetilde{x}^{\,2}\ln\widetilde{x}\right] ds + \int_0^\tau \widetilde{u}\,\widetilde{x}\,ds.$$

Thanks to (28), the well-posedness of the model (i.e. existence and uniqueness of the solution) can be investigated by using classical energy techniques based on Galerkin methods; this will be the subject of future works.

Remark 4. It can be shown Montseny [2009] that, using a change of variables on (26) given by $2\pi\zeta := \mathrm{signe}(\xi)\sqrt{|\xi|}$ and $\widetilde{\Theta}(\tau,\zeta) := \widehat{\widetilde{\psi}}\left(\tau, 4\pi^2\zeta^2\right)$, and applying inverse Fourier transform, we can state the non trivial result that the flame model is equivalent to the following one, based on the classical heat equation on $\mathbb{R}_z$:

$$\begin{cases} \partial_\tau \widetilde{\theta} = \widetilde{x}\,\partial_z^2\,\widetilde{\theta} + \widetilde{x}\,\ln(\widetilde{x}) + \widetilde{u},\ \widetilde{\theta}(0,.) = 0 \\ \widetilde{x} = 4\widetilde{\theta}\,(\tau, 0)\,, \end{cases} \tag{29}$$

where $\widetilde{\theta}(\tau,.) := \mathfrak{F}^{-1}\,\widetilde{\Theta}(\tau,.)$ and z the primal variable of ζ. This formulation also presents many interests in terms of simulation and analysis, due to properties inherited from heat equation.

5.4 Numerical results

We present in Figure 1 some numerical results obtained with the transformed model (26). As stated in Audounet et al. [1998], the flame model presents two different behavior: depending on the input u, the flame can either die ($x \to 0$) or explode ($x \to +\infty$); those two behaviors are shown on figures 1a and 1b respectively obtained with $E = 1.7$ and $E = 2$ for input $u = E(1-t)t^{0.3}$.

We compare on Figure 1c a simulation obtained with (26) and a simulation obtained with the singular model (25), at the start of the process; it highlights the singularity of the model, that can, even if it occurs at the very beginning of the simulation, be critical for many problems (identification problems for example). Moreover, the singularity generates many other problems in practice, namely the non null initial value of $x(0)$ chosen to avoid the singularity, and the fact that the results are rather sensible towards this choice. The model obtained after TST does not present any of those difficulties; in that sense, the TST allowed to "desingularize" the problem.

Finally, Figure 1d shows the correspondence $\tau \mapsto t$ in the case of flame extinction, and highlights that the time τ "follows" the system evolution: it evolves faster than time t when close to the singularity; moreover, in the case of flame extinction, the evolution of time t naturally stops, whereas a manual check is necessary when using the singular model in time t.

6. CONCLUSION

In this paper, some operatorial transformations, based on trajectorial formulations, are proposed, to simplify general non local dynamic problems. The diffusive transformation, that gives local formulations of non local models, was briefly presented. The time-scale transformations were introduced in a generic manner and their properties suggests various applications like simplification of singular non local models. The interest of such transformations was finally highlighted on a non local singular model of flame, which was simulated and whose uniqueness analysis was briefly given. Thanks to those transformations, a whole analysis of this model is possible and will be the subject of future works.

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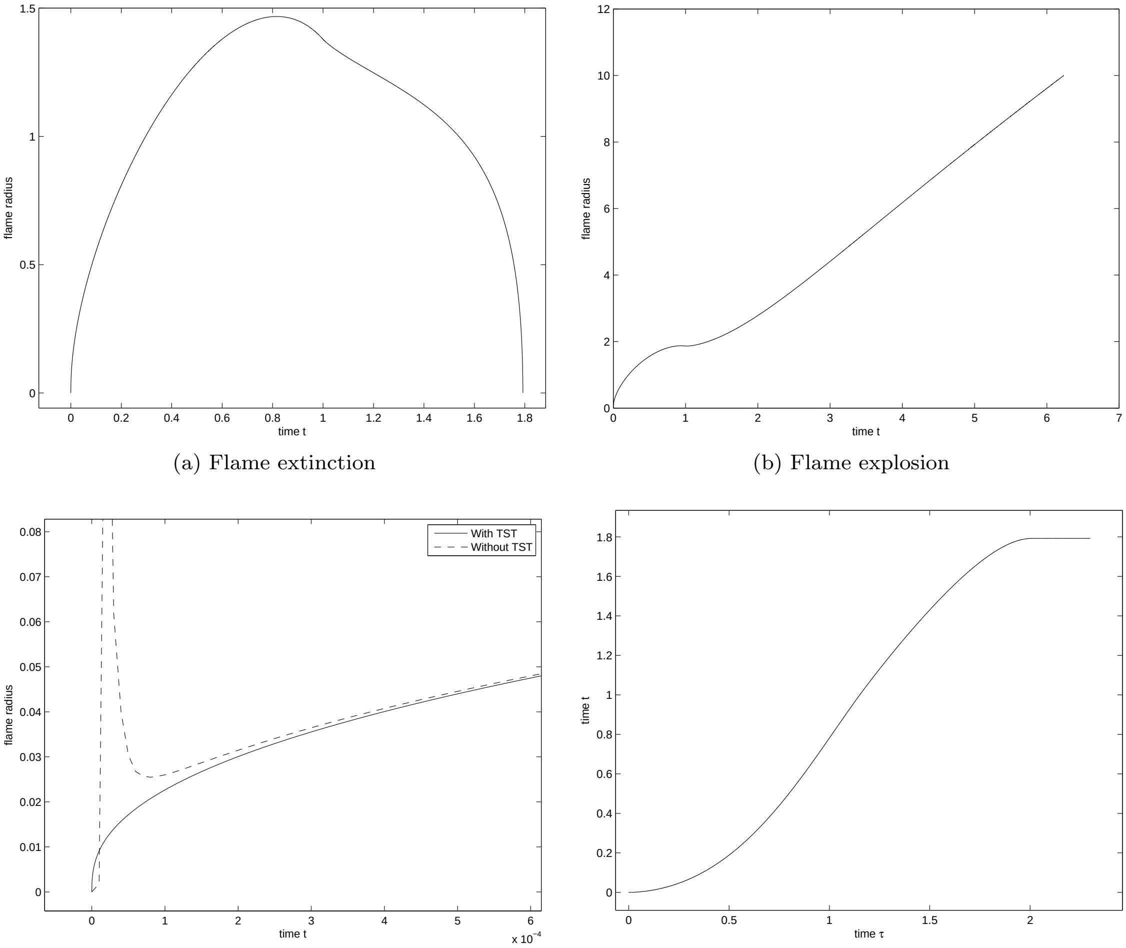

(a) Flame extinction

(b) Flame explosion

(c) Zoom at the beginning

(d) Time t according to time τ

Fig. 1. Simulations of the flame model.

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